Tilting correspondences of perfectoid towers

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This talk is based on the following:

- K. Hayashi, *Structural properties and tilting correspondences of perfectoid towers*, arxiv:2605.27283.

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Mixed characteristic $\xleftarrow{\text{Perfectoid methods}}$ Positive characteristic

A central notion is the tilting operation

$$A \rightsquigarrow A^\flat$$

for a perfectoid ring A of mixed characteristic.

Many properties/invariants correspond under $A \rightsquigarrow A^\flat$.

- Complete integral closedness.
- Adic spaces: $\mathrm{Spa}(A, A^\circ) \xrightarrow{\approx} \mathrm{Spa}(A^\flat, A^{\circ\flat})$.
- Étale cohomology: $\mathbb{R}\Gamma_{\text{ét}}(U, G) \cong \mathbb{R}\Gamma_{\text{ét}}(U^\flat, G)$.

Difficulty

Perfectoid rings (except for finite products of perfect fields of char. $p > 0$) are never Noetherian!

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$$\begin{array}{ccccccccc} R_0 & \longrightarrow & R_1 & \longrightarrow & \cdots & \longrightarrow & R_i & \longrightarrow & \cdots \longrightarrow \widehat{R_\infty} \\ & & & & & & \Big\downarrow \text{(small) tilting} & & \Big\downarrow \text{tilting} \\ R_0^{s.\flat} & \longrightarrow & R_1^{s.\flat} & \longrightarrow & \cdots & \longrightarrow & R_i^{s.\flat} & \longrightarrow & \cdots \longrightarrow \widehat{R_\infty^{s.\flat}} \end{array}$$

Toy example

$$\begin{array}{ccccccccc} \mathbb{Z}_p & \subset & \mathbb{Z}_p[p^{1/p}] & \subset & \cdots & \subset & \mathbb{Z}_p[p^{1/p^i}] & \subset & \cdots \subset \widehat{\mathbb{Z}_p[p^{1/p^\infty}]} \\ & & & & & & \Big\downarrow & & \Big\downarrow \\ \mathbb{F}_p[\![T]\!] & \subset & \mathbb{F}_p[\![T^{1/p}]\!] & \subset & \cdots & \subset & \mathbb{F}_p[\![T^{1/p^i}]\!] & \subset & \cdots \subset \widehat{\mathbb{F}_p[\![T^{1/p^\infty}]\!]} \end{array}$$

Question

What kind of properties/invariants correspond under $R_i \rightsquigarrow R_i^{s.\flat}$?

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Perfectoid towers

Definition 1 (Ishiro–Nakazato–Shimomoto[1])

p : a prime number. R : a ring. A perfectoid tower (arising from (R, p)) is an inductive system of rings $R = R_0 \to R_1 \to \cdots \to R_i \xrightarrow{t_i} \cdots$ s.t.

1. Every R_i is p-adically complete (for simplicity).
2. $^{\exists}\pi \in R_1$ with $\pi^p R_1 = pR_1$ s.t. $^{\forall}i \geq 0$:

$$\begin{array}{ccc} R_{i+1}/(p) & \xrightarrow{\mathrm{Frob}} & R_{i+1}/(p) \\ \downarrow & \overset{F_i}{\searrow} & \uparrow{\scriptstyle \overline{t_i}} \\ R_{i+1}/(\pi) & \xrightarrow{\cong} & R_i/(p) \end{array}$$

3. $\mathrm{Ann}_{R_i}(p) = \bigcup_{n \geq 0} \mathrm{Ann}_{R_i}(p^n)$ and $\mathrm{Ann}_{R_{i+1}}(p) \overset{\cong}{\dashrightarrow} \mathrm{Ann}_{R_i}(p)$ $(^{\forall}i)$.

- $\widehat{R_\infty} := \left(\varinjlim_i R_i\right)^{\wedge_p}$ is a perfectoid ring.

[1] Algebra Number Theory **19**(12), (2025), 2307–2358.

The i-th small tilt is

$$R_i^{s.\flat} \overset{\text{def}}{=} \varprojlim \left(R_i/pR_i \xleftarrow{F_i} R_{i+1}/pR_{i+1} \xleftarrow{F_{i+1}} R_{i+2}/pR_{i+2} \xleftarrow{F_{i+2}} \cdots \right).$$

- $\{R_i^{s.\flat}\}_{i\geq 0}$ is a perfect(oid) tower ($R_0^{s.\flat}$ is reduced & $R_i^{s.\flat} \cong (R_0^{s.\flat})^{\frac{1}{p^i}}$).

Example 2

R. Ishizuka[2] gave a construction from Frobenius lifts.
Consequently, $^{\exists}$ a perfectoid tower arising from the following R:

- Complete local (log-)regular rings.
- Affine semigroup rings $W(k)[\![H]\!]$ $(H \subset \mathbb{Z}^n) \rightsquigarrow k[\![H]\!][\![T]\!]$.
- (Ishiro–Shimomoto[3]) p-Stanley–Reisner rings

$$W(k)\overline{[\![\Delta]\!]} = \frac{W(k)[\![x_2,\ldots,x_n]\!]}{(\text{squarefree monomials in } p, x_2, \ldots, x_n)} \rightsquigarrow k[\![\Delta]\!].$$

- (H.[4]) Rational double points over $W(k)$ $(k = \overline{k}) \rightsquigarrow$ RDPs over k.

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Theorem 3 (Ishiro–Nakazato–Shimomoto)

$^{\exists}p^{s.\flat} \in R_0^{s.\flat}$ s.t. $^{\forall}i \geq 0$, $R_i^{s.\flat}$ is $p^{s.\flat}$-adically complete and

1. $R_i^{s.\flat}/(p^{s.\flat}) \xrightarrow{\cong} R_i/(p)$.
2. $\mathrm{Ann}_{R_i^{s.\flat}}(p^{s.\flat}) \xrightarrow{\cong} \mathrm{Ann}_{R_i}(p)$.

Example 4

$\mathbb{F}_p[\![T^{1/p^i}]\!]/(T) \xrightarrow{\cong} \mathbb{F}_p[T]/(T^{p^i}) \cong \mathbb{Z}_p[p^{1/p^i}]/(p)$.

Corollary 5

1. Fix $i \geq 0$. R_i is Noetherian (resp. local) $\iff$ so is $R_i^{s.\flat}$.
2. Fix $i \geq 0$. p is R_i-regular $\iff$ $p^{s.\flat}$ is $R_i^{s.\flat}$-regular.

Assume R_i is Noetherian local. If p is R_i-regular, Theorem 3 ① implies

R_i is CM (resp. Gorenstein, complete intersection) $\iff$ so is $R_i^{s.\flat}$.

We will consider more precise correspondences in the general case.

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Main results

$\{R_i\}_{i\geq 0}$: a perfectoid tower.

Theorem A (Canonical decomposition)

1. $\{S_i := R_i/\operatorname{Ann}(p)\}_{i\geq 0}$ is a perfectoid tower. $S_i^{s.\flat} = R_i^{s.\flat}/\operatorname{Ann}(p^{s.\flat})$.
2. One has the cartesian diagrams of rings

$$\begin{array}{ccccccc} R_i & \longrightarrow & R_i/(p) & \xleftarrow{\cong} & R_i^{s.\flat}/(p^{s.\flat}) & \longleftarrow & R_i^{s.\flat} \\ \downarrow & \square & \downarrow & & \downarrow & \square & \downarrow \\ S_i & \longrightarrow & S_i/(p) & \xleftarrow{\cong} & S_i^{s.\flat}/(p^{s.\flat}) & \longleftarrow & R_i^{s.\flat}/(p^{s.\flat}). \end{array}$$

$\rightsquigarrow$ One can reduce many questions about $\{R_i\}_{i\geq 0}$
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Remark

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Assume $\boxed{(R_i, \mathfrak{m}_i) : \text{complete Noetherian local \& } k = R_0/\mathfrak{m}_0 : \text{perfect}}$ and $\boxed{\text{fix } i \geq 0.}$

Theorem B (Koszul complexes)

Let $\boxed{\boldsymbol{x} = p, x_2, \ldots, x_n \subset R_i \text{ and } \boldsymbol{x}^{s.\flat} = p^{s.\flat}, x_2^{s.\flat}, \ldots, x_n^{s.\flat} \subset R_i^{s.\flat}}$ be sequences of elements corresponding under $R_i^{s.\flat}/(p^{s.\flat}) \xrightarrow{\cong} R_i/(p)$. Then

$$K_\bullet(\boldsymbol{x}; R_i) \cong K_\bullet(\boldsymbol{x}^{s.\flat}; R_i^{s.\flat}) \quad \text{in} \quad D(\operatorname{Mod} W(k)).$$

By Theorem A, we may reduce to the case where p is R_i-regular. Then

$$K_\bullet(\boldsymbol{x}; R_i) = K_\bullet(p; R_i) \otimes K_\bullet(\boldsymbol{x}'; R_i) \xrightarrow{\cong} K_\bullet(\boldsymbol{x}'; R_i/(p)),$$

$$K_\bullet(\boldsymbol{x}^{s.\flat}; R_i^{s.\flat}) = K_\bullet(p^{s.\flat}; R_i^{s.\flat}) \otimes K_\bullet(\boldsymbol{x}^{s.\flat\prime}; R_i^{s.\flat}) \xrightarrow{\cong} K_\bullet(\boldsymbol{x}^{s.\flat\prime}; R_i^{s.\flat}/(p^{s.\flat})),$$

where $\boldsymbol{x}' := x_2, \ldots, x_n$ and $\boldsymbol{x}^{s.\flat\prime} := x_2^{s.\flat}, \ldots, x_n^{s.\flat}$. □

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Theorem C

1. $\dim(R_i) = \dim(R_i^{s.b})$.

$$\dim R_i = \max\{\dim S_i, \dim \frac{R_i}{(p)}\} = \max\{\dim S_i^{s.b}, \dim \frac{R_i^{s.b}}{(p^{s.b})}\} = \dim R_i^{s.b}. \square$$

2. $\operatorname{edim}(R_i) = \operatorname{edim}(R_i^{s.b})$. Hence R_i is regular $\iff$ so is $R_i^{s.b}$.

$$\operatorname{edim}(R_i) = \begin{cases} \operatorname{edim} R_i/(p) & (p \in \mathfrak{m}_i^2), \\ \operatorname{edim} R_i/(p) + 1 & (p \notin \mathfrak{m}_i^2) \end{cases}$$

$$\overset{\text{Check!}}{=} \begin{cases} \operatorname{edim} R_i^{s.b}/(p^{s.b}) & (p^{s.b} \in (\mathfrak{m}_i^{s.b})^2), \\ \operatorname{edim} R_i^{s.b}/(p^{s.b}) + 1 & (p^{s.b} \notin (\mathfrak{m}_i^{s.b})^2) \end{cases} = \operatorname{edim}(R_i^{s.b}). \quad \square$$

3. $\operatorname{depth}(R_i) = \operatorname{depth}(R_i^{s.b})$. Hence R_i is CM $\iff$ so is $R_i^{s.b}$.

Take $\boldsymbol{x}$ and $\boldsymbol{x}^{s.b}$ s.t. $\mathfrak{m}_i = (\boldsymbol{x})$ and $\mathfrak{m}_i^{s.b} = (\boldsymbol{x}^{s.b})$. Then

$$\operatorname{depth}(R_i) = n - \max\{q \mid H_q(\boldsymbol{x}) \neq 0\} \qquad \text{(Depth sensitivity)}$$

$$\overset{\text{Thm. B}}{=} n - \max\{q \mid H_q(\boldsymbol{x}^{s.b}) \neq 0\} = \operatorname{depth}(R_i^{s.b}). \quad \square$$

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5. R_i is CM of type r (resp. Gorenstein) $\iff$ so is $R_i^{s.\flat}$.

$\operatorname{edim}(R_i) = \operatorname{edim}(R_i^{s.\flat})$ allows us to choose $\boldsymbol{x}$ and $\boldsymbol{x}^{s.\flat}$ as minimal basis. May assume R_i and $R_i^{s.\flat}$ are CM. Then

$$r(R_i) = \dim_k H_c(\boldsymbol{x}) \overset{\text{Thm. B}}{=} \dim_k H_c(\boldsymbol{x}^{s.\flat}) = r(R_i^{s.\flat}).$$ □

6. R_i is complete intersection of codimension c $\iff$ so is $R_i^{s.\flat}$.

LHS $\iff \dim_k H_1(\boldsymbol{x}) = c \overset{\text{Thm. B}}{\iff} \dim_k H_1(\boldsymbol{x}^{s.\flat}) = c \iff$ RHS. □

Example 6

$R = W(k)\overline{[\![\Delta]\!]}$ is CM (resp. Gor.) $\iff$ so is $R^{s.\flat} = k[\![\Delta]\!]$. Hence the CM and Gor. properties can be characterized in terms of Δ, even in mixed char.

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LHS $\iff \dim_k H_1(\boldsymbol{x}) = c \overset{\text{Thm. B}}{\iff} \dim_k H_1(\boldsymbol{x}^{s.\flat}) = c \iff$ RHS. □

Example 6

$R = W(k)\overline{[\![\Delta]\!]}$ is CM (resp. Gor.) $\iff$ so is $R^{s.\flat} = k[\![\Delta]\!]$. Hence the CM and Gor. properties can be characterized in terms of Δ, even in mixed char.

4. $\operatorname{codim}(R_i) = \operatorname{codim}(R_i^{s.\flat}) \boxed{=: c}$.

$$\begin{aligned}\operatorname{codim}(R_i) &= \operatorname{edim}(R_i) - \operatorname{depth}(R_i) \\ &= \operatorname{edim}(R_i^{s.\flat}) - \operatorname{depth}(R_i^{s.\flat}) = \operatorname{codim}(R_i^{s.\flat}).\end{aligned}$$ □

5. R_i is CM of type r (resp. Gorenstein) $\iff$ so is $R_i^{s.\flat}$.

$\operatorname{edim}(R_i) = \operatorname{edim}(R_i^{s.\flat})$ allows us to **choose $\boldsymbol{x}$ and $\boldsymbol{x}^{s.\flat}$ as minimal basis.**
May assume R_i and $R_i^{s.\flat}$ are CM. Then

$$r(R_i) = \dim_k H_c(\boldsymbol{x}) \overset{\text{Thm. B}}{=} \dim_k H_c(\boldsymbol{x}^{s.\flat}) = r(R_i^{s.\flat}).$$ □

6. R_i is complete intersection of codimension $c \iff$ so is $R_i^{s.\flat}$.

LHS $\iff \dim_k H_1(\boldsymbol{x}) = c \overset{\text{Thm. B}}{\iff} \dim_k H_1(\boldsymbol{x}^{s.\flat}) = c \iff$ RHS. □

Example 6

$R = W(k)\overline{[\![\Delta]\!]}$ is CM (resp. Gor.) $\iff$ so is $R^{s.\flat} = k[\![\Delta]\!]$. Hence the CM and Gor. properties can be characterized in terms of Δ, even in mixed char.

4. $\operatorname{codim}(R_i) = \operatorname{codim}(R_i^{s.\flat}) \boxed{=: c}$.

$$\begin{aligned}\operatorname{codim}(R_i) &= \operatorname{edim}(R_i) - \operatorname{depth}(R_i) \\ &= \operatorname{edim}(R_i^{s.\flat}) - \operatorname{depth}(R_i^{s.\flat}) = \operatorname{codim}(R_i^{s.\flat}).\end{aligned}$$ □

5. R_i is CM of type r (resp. Gorenstein) $\iff$ so is $R_i^{s.\flat}$.

$\operatorname{edim}(R_i) = \operatorname{edim}(R_i^{s.\flat})$ allows us to $\boxed{\text{choose } \boldsymbol{x} \text{ and } \boldsymbol{x}^{s.\flat} \text{ as minimal basis.}}$
May assume R_i and $R_i^{s.\flat}$ are CM. Then

$$r(R_i) = \dim_k H_c(\boldsymbol{x}) \overset{\text{Thm. B}}{=} \dim_k H_c(\boldsymbol{x}^{s.\flat}) = r(R_i^{s.\flat}).$$ □

6. R_i is complete intersection of codimension c $\iff$ so is $R_i^{s.\flat}$.

LHS $\iff \dim_k H_1(\boldsymbol{x}) = c \overset{\text{Thm. B}}{\iff} \dim_k H_1(\boldsymbol{x}^{s.\flat}) = c \iff$ RHS. □

Example 6

$R = W(k)\overline{[\![\Delta]\!]}$ is CM (resp. Gor.) $\iff$ so is $R^{s.\flat} = k[\![\Delta]\!]$. Hence the CM and Gor. properties can be characterized in terms of Δ, even in mixed char.