Regular rings and perfectoid towers

Kazuki HAYASHI

Institute of Science Tokyo

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This talk is based on the following:

- K. Hayashi, *Regular rings and perfectoid towers*, arXiv:2605.27337.

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Kunz's theorem[1]

For a Noetherian $\mathbb{F}_p$-algebra R, TFAE.

1. R is regular.
2. The absolute Frobenius $\varphi\colon R \to R$ is (faithfully) flat.
3. The perfect R-algebra $R_{\mathrm{perf}} := \varinjlim \left(R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots \right)$ is faithfully flat.
4. There exists a perfect R-algebra A that is faithfully flat.

- An $\mathbb{F}_p$-algebra A is perfect $\overset{\text{def}}{\iff} \varphi\colon A \to A$ is bijective.
- For ④ ⇒ ②, observe the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\text{faithfully flat}"'] & R \arrow[d, "\text{(faithfully) flat}"] \\ A \arrow[r, "\cong", "\varphi"'] & A \end{tikzcd}$$

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Perfectoid algebras are generalizations of perfect $\mathbb{F}_p$-alg. to mixed char.

p-adic Kunz's theorem (B. Bhatt–S. B. Iyengar–L. Ma[2])

For a Noetherian ring R such that $p \in \mathrm{rad}(R)$, TFAE.

1. R is regular.
2. There exists a perfectoid R-algebra A that is faithfully flat.

e.g., $R = \mathbb{Z}_p \to A = \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]} = \varinjlim \left(\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[p^{\frac{1}{p}}] \hookrightarrow \mathbb{Z}_p[p^{\frac{1}{p^2}}] \hookrightarrow \cdots\right)^{\wedge}$.

A ring A is perfectoid (w.r.t. a fixed prime p) $\iff {}^{\exists}\varpi \in A$ s.t.

1. $p \in \varpi^p A$, and A is ϖ-adically complete.
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$$\begin{array}{ccc} A/\varpi^p A & \xrightarrow{\varphi} & A/\varpi^p A \\ \downarrow & & \| \\ A/\varpi A & \overset{\cong}{\dashrightarrow} & A/\varpi^p A \end{array}$$

3. $A_{\varpi\text{-tor}} \xrightarrow{\sim} A_{\varpi\text{-tor}};\ x \mapsto x^p$. Here $A_{\varpi\text{-tor}} := \bigcup_{n>0}(0 :_A \varpi^n)$.

[2] Comm. Algebra **47**(6), (2019), 2367–2383.

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Definition (S. Ishiro–K. Nakazato–K. Shimomoto[3])

An inductive system $\{R_i\}_{i \geq 0} = (R_0 \to R_1 \to \cdots \to R_i \xrightarrow{t_i} \cdots)$ of Noetherian rings indexed by $\mathbb{N}$ is a (Noetherian) perfectoid tower (arising from (R_0, p)) $\overset{\text{def}}{\iff}$ ${}^\exists \varpi \in R_1$ s.t.

1. $pR_1 = \varpi^p R_1$, and $p \in \operatorname{rad}(R_i)$.
2. $$\begin{tikzcd} R_{i+1}/pR_{i+1} \arrow[r, "\varphi"] \arrow[d] \arrow[dr, dashed, "{}^\exists F_i"] & R_{i+1}/pR_{i+1} \\ R_{i+1}/\varpi R_{i+1} \arrow[r, dashed, "\cong"] & R_i/pR_i \arrow[u, hook, "\overline{t_i}"'] \end{tikzcd}$$
3. $p(R_i)_{p\text{-tor}} = 0$, and F_i induces $(R_{i+1})_{p\text{-tor}} \xrightarrow{\sim} (R_i)_{p\text{-tor}}$.

$\Longrightarrow \left(\varinjlim_i R_i\right)^\wedge_p$ is a perfectoid ring.

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To a perfectoid tower $\{R_i\}_{i\geq 0}$, we associate the perfectoid tower of $\mathbb{F}_p$-algebras $\{R_i^{s.\flat}\}_{i\geq 0}$ (called the tilt of $\{R_i\}_{i\geq 0}$) defined by

$$R_i^{s.\flat} \stackrel{\text{def}}{=} \varprojlim \left(R_i/pR_i \xleftarrow{F_i} R_{i+1}/pR_{i+1} \xleftarrow{F_{i+1}} R_{i+2}/pR_{i+2} \xleftarrow{F_{i+2}} \cdots \right).$$

Examples

1. $\{R_i\}_{i\geq 0} = \{\mathbb{Z}_p[p^{1/p^i}]\}_{i\geq 0} \rightsquigarrow \{R_i^{s.\flat}\}_{i\geq 0} = \{\mathbb{F}_p[\![T]\!][T^{1/p^i}]\}_{i\geq 0}$.
2. A perfectoid tower of $\mathbb{F}_p$-algebras is precisely a perfect tower, i.e., is isomorphic to $(R \xrightarrow{\varphi} R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots)$ for some Noetherian $\mathbb{F}_p$-alg. R.
3. R. Ishizuka and Ishiro–Shimomoto constructed from:
   - affine semigroup rings $\mathbb{Z}_p[\![H]\!]$ $(H \subset \mathbb{N}^{\oplus n})$.

$$\mathbb{Z}_p[\![t^2, t^3]\!] \hookrightarrow \mathbb{Z}_p[p^{1/p}][\![t^{2/p}, t^{3/p}]\!] \hookrightarrow \mathbb{Z}_p[p^{1/p^2}][\![t^{2/p^2}, t^{3/p^2}]\!] \hookrightarrow \cdots,$$
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   - p-Stanley–Reisner rings $\mathbb{Z}_p[\![x_2, \ldots, x_n]\!]/(\square\text{free monomials in } p, x_2, \ldots, x_n)$.

$$\mathbb{Z}_p[\![x, y]\!]/(px, py) \hookrightarrow \mathbb{Z}_p[p^{1/p}][\![x^{1/p}, y^{1/p}]\!]/(p^{1/p}x^{1/p}, p^{1/p}y^{1/p}) \hookrightarrow \cdots$$
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Theorem A

$\{R_i\}_{i\geq 0}$: a perfectoid tower ($\rightsquigarrow \{R_i^{s.\flat}\}_{i\geq 0}$). Fix $i \geq 0$. Then TFAE.

1. R_i is regular.
2. $t_i \colon R_i \to R_{i+1}$ is (faithfully) flat.
3. $R_i^{s.\flat}$ ($\cong R_0^{s.\flat}$) is regular.
4. $t_i^{s.\flat} \colon R_i^{s.\flat} \to R_{i+1}^{s.\flat}$ ($\cong \varphi \colon R_0^{s.\flat} \to R_0^{s.\flat}$) is (faithfully) flat.

In particular, both ① and ② are independent of i.

Corollary B (Rigidity of regularity)

R_i is regular for **some** $i \implies R_i$ is regular for **any** $i \geq 0$.

Corollary C (Kunz-type theorem in terms of perfectoid towers)

For a Noetherian local ring $(R, \mathfrak{m})$ such that $p \in \mathfrak{m}$, TFAE.

1. R is regular.
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Corollary C gives another proof of the following result without the regularity criterion of Auslander–Buchsbaum–Serre.

Corollary D

If $(R, \mathfrak{m})$ is a regular local ring such that $p \in \mathfrak{m}$, then $R_{\mathfrak{p}}$ is regular for every $\mathfrak{p} \in V(pR)$.

- Ishizuka–Nakazato[4] also gave another proof using prismatic Kunz's theorem. While their proof relies on the methods from homotopy theory, our proof is based on a more elementary argument.

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Theorem A (recall)

Fix $i \geq 0$. Then TFAE.

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Sketch of proof We may assume that R_i is local.

③ ⇔ ④: Kunz's theorem.

① ⇒ ②: Show "R_{i+1} is CM" & "$\dim(R_{i+1}) = \dim(R_i)$."

② ⇔ ④: Use the following results of Ishiro–Nakazato–Shimomoto:

- $R_i^{s.\flat}/p^{s.\flat}R_i^{s.\flat} \xrightarrow{\cong} R_i/pR_i$ (e.g., $\mathbb{F}_p[\![T]\!]/(T) \xrightarrow{\cong} \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$).
- p is R_i-regular $\Longleftrightarrow p^{s.\flat}$ is $R_i^{s.\flat}$-regular. $\cdots (*)$

The case $(*)$ follows from the local criterion of flatness. Reduce the general case to $(*)$ via $\widetilde{R_i} := R_i/(R_i)_{p\text{-tor}}$ ($\rightsquigarrow (\widetilde{R_i})^{s.\flat} \cong R_i^{s.\flat}/(R_i^{s.\flat})_{p^{s.\flat}\text{-tor}}$).

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Sketch of proof We may assume that R_i is local.

③ ⇔ ④: Kunz's theorem.

① ⇒ ②: Show "R_{i+1} is CM" & "$\dim(R_{i+1}) = \dim(R_i)$."

② ⇔ ④: Use the following results of Ishiro–Nakazato–Shimomoto:

- $R_i^{s.\flat}/p^{s.\flat}R_i^{s.\flat} \xrightarrow{\cong} R_i/pR_i$ (e.g., $\mathbb{F}_p[\![T]\!]/(T) \xrightarrow{\cong} \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$).
- p is R_i-regular $\Longleftrightarrow p^{s.\flat}$ is $R_i^{s.\flat}$-regular. $\cdots(*)$

The case $(*)$ follows from the local criterion of flatness. Reduce the general case to $(*)$ via $\widetilde{R_i} := R_i/(R_i)_{p\text{-tor}}$ ($\rightsquigarrow (\widetilde{R_i})^{s.\flat} \cong R_i^{s.\flat}/(R_i^{s.\flat})_{p^{s.\flat}\text{-tor}}$).

Theorem A (recall)

Fix $i \geq 0$. Then TFAE.

1. R_i is regular.
2. $t_i\colon R_i \to R_{i+1}$ is (faithfully) flat.
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② + ③ ⇒ ①: Suggested by O. Gabber, J. Lurie[5] proved the following.

A mixed characteristic analogue of Kunz's theorem

$(R, \mathfrak{m})$: Noetherian local. $\varpi \in \mathfrak{m}$: R-regular with $p \in \varpi^p R$. TFAE.

1. R is regular.
2. The homomorphism $\psi\colon R/\varpi R \to R/\varpi^p R$; $x \mapsto x^p$ is (faithfully) flat.

③ $\Longrightarrow p^{s.\flat}$ is $R_i^{s.\flat}$-regular $\Longleftrightarrow p$ is R_i-regular $\Longleftrightarrow \varpi$ is R_{i+1}-regular.
Hence one can apply Theorem above to R_{i+1}. Observe:

$$\begin{tikzcd}
R_{i+1}/pR_{i+1} \arrow[r, "\varphi"] \arrow[d] & R_{i+1}/pR_{i+1} \\
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\end{tikzcd}$$

Thus R_{i+1} is regular, and since t_i is faithfully flat, R_i is regular. □

[5] p-Adic Hodge Theory, Singular Varieties, and NonAbelian Aspects, Simons Symposia, Springer International Publishing, (2023), 239–252.

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Summary

1. (Kunz '69) TFAE for $R \supset \mathbb{F}_p$.
   - R is regular.
   - $\varphi\colon R \to R$ is (faithfully) flat.
   - There exists a perfect R-algebra A that is faithfully flat.
2. (Bhatt–Iyengar–Ma '19) TFAE for R with $p \in \mathrm{rad}(R)$.
   - R is regular.
   - There exists a perfectoid R-algebra A that is faithfully flat.
3. (Gabber–Lurie '23) TFAE for $(R, \mathfrak{m})$ with $\varpi \in \mathrm{NZD}(R)$ & $p \in \varpi^p R$.
   - R is regular.
   - $R/\varpi R \to R/\varpi^p R$; $x \mapsto x^p$ is (faithfully) flat.
4. (H. '26) TFAE for $(R, \mathfrak{m})$ with $p \in \mathfrak{m}$.
   - R is regular.
   - There exists a perfectoid tower starting from R whose i-th transition map is (faithfully) flat for any (or, equivalently, some) $i \geq 0$.