パーフェクトイド塔の理論について

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§1. Definitions & Examples.

§2. Tilting correspondences.

Fix a prime number p

§1. Definitions & Examples.

char. p

perfect ring

$\mathbb{F}_p[T^{\frac{1}{p^\infty}}]$

generalization $\longrightarrow$

mixed char.

perfectoid ring

$\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$

perfect tower $\longrightarrow$ perfectoid tower

$R \hookrightarrow R^{\frac{1}{p}} \hookrightarrow R^{\frac{1}{p^2}} \hookrightarrow \cdots$

(R: reduced)

Def (Ishiro-Nakazato-Shimomoto)

R : a comm. ring

$p \in I_0 \subset R$: a principal ideal

A perfectoid tower arising from (R, I_0) is an inductive system of rings

$$\{R_i\}_{i \geq 0}: \quad R = R_0 \hookrightarrow R_1 \hookrightarrow R_2 \hookrightarrow \cdots$$

s.t. ① R_i is I_0-adically complete (for simplicity)

② $\exists I_1 \subset R_1$: principal with $I_1^p = I_0 R_1$

& $\forall i$

$$\begin{array}{ccc} R_{i+1}/I_0 R_{i+1} & \xrightarrow{\varphi} & R_{i+1}/I_0 R_{i+1} \\ \downarrow & \overset{\exists!}{\searrow^{F_i}} & \uparrow \\ R_{i+1}/I_1 R_{i+1} & \cong & R_i/I_0 R_i \end{array}$$

③ $\forall i$, $R_i[I_0^\infty] = R_i[I_0]$

& $R_{i+1}[I_0] \xrightarrow{\cong} R_i[I_0]$

$$\begin{array}{ccc} \downarrow & \curvearrowright & \updownarrow \\ R_{i+1}/I_0 R_{i+1} & \xrightarrow[F_i]{} & R_i/I_0 R_i \end{array}$$

- $\widehat{R_\infty} := \left(\varinjlim_i R_i\right)^\wedge_{I_0}$ is a perfectoid ring.

- $R_i^{s.b} := \varprojlim \left(R_i/I_0R_i \xleftarrow{F_i} R_{i+1}/I_0R_{i+1} \xleftarrow{F_{i+1}} R_{i+2}/I_0R_{i+2} \leftarrow \cdots \right)$

  i^{th} small tilt

$$\boxed{\{R_i\}_{i\geq 0} \overset{\text{tilting}}{\rightsquigarrow} \{R_i^{s.b}\}_{i\geq 0}}$$

Ex ① $\mathbb{Q}_p \hookrightarrow \mathbb{Q}_p[p^{\frac{1}{p}}] \hookrightarrow \mathbb{Q}_p[p^{\frac{1}{p^2}}] \hookrightarrow \cdots$

$\wr$

$\mathbb{F}_p[\![T]\!] \hookrightarrow \mathbb{F}_p[\![T^{\frac{1}{p}}]\!] \hookrightarrow \mathbb{F}_p[\![T^{\frac{1}{p^2}}]\!] \hookrightarrow \cdots$

$\wr$

② $\mathbb{Q}_p[\zeta_p] \hookrightarrow \mathbb{Q}_p[\zeta_{p^2}] \hookrightarrow \mathbb{Q}_p[\zeta_{p^3}] \hookrightarrow \cdots$

③ Ishizuka gave a construction from Frobenius lifts

$\rightarrow$ $\exists$ a perfectoid tower arising from $(R, (p))$, where

- R : complete local log-regular ring
- $R = \bigoplus_{n>0} H^0(\widetilde{A}, \widetilde{\mathcal{L}})$

  ↑ the canonical lift of an ordinary abelian variety $/k$

  $\widetilde{\mathcal{L}}$: ample line bdl.

Problem 1 Find other examples/constructions.

Thm A (H.-Ishiro-Shimomoto)

$R = W(k)[\![x_2, \cdots, x_d]\!]$

$\to \left\{R_i := R[p^{\frac{1}{p^i}}, x_2^{\frac{1}{p^i}}, \cdots, x_d^{\frac{1}{p^i}}]\right\}_{i \geq 0}$: a perfectoid tower

$R \hookrightarrow S$: finite ext. of normal local domains

s.t. $R[\frac{1}{p}] \to S[\frac{1}{p}]$ is étale.

$S_i :=$ the normalization of $R_i \otimes_R S$.

$\Rightarrow \exists N \gg 0 \ \exists \varepsilon \in (0,1) \cap \mathbb{Q}$ s.t.

$S_N \to S_{N+1} \to S_{N+2} \to \cdots$

is a perfectoid tower arising from $(S_N, (p^\varepsilon))$.

Its proof is based on ramification theory

(Faltings, Andreatta)

ε comes from:

$v_p(\text{discriminant of } S_{n,\mathfrak{P}_n}/R_{n,\mathfrak{P}_n}) \searrow 0$ as $n \to \infty$

$\exists! \ \mathfrak{P}_n \mid p$

$\cap$

R_n

Conj $\forall R$: complete Noetherian local domain of mixed char.

$\exists$ a perfectoid tower arising from $(R, \exists I_0)$?

Rem ① Yamaguchi gave counter-examples to "$I_0 = (p)$"

② Open AI proposed a counter-example.

(hexagon : 2610.00465v1)

§3. Tilting correspondences

$R_0 \to R_1 \to R_2 \to \cdots$: a perfectoid tower

$\cup$

$I_0 = (f_0)$

Thm (Ishiro-Nakazato-Shimomoto)

① $\exists f_0^{s.b} \in R_0^{s.b}$ s.t. $R_i^{s.b}/(f_0^{s.b}) \xrightarrow{\cong} R_i/(f_0)$

② $R_i^{s.b}[f_0^{s.b}] \xrightarrow{\cong} R_i[f_0]$

③ Under some normality assumption,

$$\#H^q_{\text{ét}}(U; \mathbb{Z}/\ell^n\mathbb{Z}) \le \#H^q_{\text{ét}}(U^{s.b}; \mathbb{Z}/\ell^n\mathbb{Z})$$

where $\operatorname{Spec} R_i\left[\frac{1}{f_0}\right] \subset U \subset \operatorname{Spec} R_i$

$\updownarrow$ ①

$\operatorname{Spec} R_i^{s.b}\left[\frac{1}{f_0^{s.b}}\right] \subset U^{s.b} \subset \operatorname{Spec} R_i^{s.b}$

They deduced from ③ that for R: log-regular,

$Cl(R)_{\text{tors}} \otimes \mathbb{Z}[\frac{1}{p}]$ is finite.

Problem 2 Find other tilting correspondences & applications

Thm B $\underline{x} = f_0, x_2, \cdots, x_n \in R_i$

$\updownarrow$ Thm 1 ①

$\underline{x}^{s.b} = f_0^{s.b}, x_2^{s.b}, \cdots, x_n^{s.b} \in R_i^{s.b}$

$\Rightarrow \exists$ a quasi-isom. of Koszul complexes

$$K_\bullet(\underline{x}; R_i) \cong K_\bullet(\underline{x}^{s.b}; R_i^{s.b})$$

Rem $n=1 \iff$ Thm 3 ① ②

$$K_\bullet(f_0; R_i) = \left[R_i \xrightarrow{f_0} R_i\right]$$

Assume R_i is Noetherian local

$\left(\underset{\text{Thm 1 ①}}{\iff} R_i^{s.b}\right.$ is so. $)$

Thm C R_i is regular $\iff R_i^{s.b}$ is so.

$\Downarrow$

(resp. complete intersection,

$\Downarrow$

Gorenstein,

$\Downarrow$

Cohen-Macaulay)