Gabber-LurieによるKunzの混標数版定理とそのパーフェクトイド塔への応用

- Fix a prime number p.
- $(R, \mathfrak{m}, k)$: a comm. Noeth. local ring.

Thm 1 (Kunz '69)

Assume $R \supset \mathbb{F}_p$

Then R is regular $\Leftrightarrow$ $\varphi : R \to R$ is flat

$$\begin{array}{ccc} R & \to & R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \mapsto & x^p \end{array}$$

Thm 2 (Gabber-Lurie '23)

Assume $p \in \mathfrak{m}$

and $\exists \pi \in R$ s.t. $p \in \pi^p R$

① R is regular $\Rightarrow$ $\varphi : R/\pi R \to R/\pi^p R$ is flat

$$\begin{array}{ccc} R/\pi R & \to & R/\pi^p R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & x^p \end{array}$$

② "$\Leftarrow$" holds if $\pi \in NZD(R)$.

J.Lurie, *Full level structures on elliptic curves*,
in: p-Adic Hodge Theory, Singular Varieties, and Non-Abelian Aspects, Simons Symposia, Springer International Publishing, 2023, pp. 239--252.

Ex 3

① $R = \mathbb{Z}_p[[T]][p^{\frac{1}{p}}] \xleftarrow{\cong} \mathbb{Z}_p[[T,U]]/(U^p - p)$

$\pi = p^{\frac{1}{p}}$

$\Rightarrow \varphi : \mathbb{F}_p[[T]] \to \mathbb{F}_p[[T,U]]/(U^p)$

has a free basis

$\{T^\alpha U^\beta \mid 0 \leq \alpha, \beta \leq p-1\}$.

② $R = \mathbb{F}_p[[T]]/(T^p)$

$\pi = T$

$\Rightarrow \varphi : \mathbb{F}_p \to R$ is flat,

but R is not regular.

In the talk, Ryo Takahashi asked:

$$\begin{array}{rcl} \varphi : R/\pi R & \to & R/\pi^p R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \mapsto & x^p \end{array}$$

is flat $\overset{?}{\Rightarrow}$ $R/\pi R$ is regular.

A. No!

$R = \mathbb{F}_p[[T,U]]/(T^p, U^p)$ is a counterexample.

$\rotatebox{90}{$\in$}$

$\pi = T$

Proof of Thm 2

① <u>WMA</u> • R is complete
• k is perfect

By Cohen's structure thm.,

$$R \xleftarrow{\cong} S/(p-f),$$

where • $S := W(k)[[X_1,\cdots,X_d]]$

• $f = X_1$ or $f \in (X_1,\cdots,X_d)^2$

Consider $\phi : S \to S$, $X_i \mapsto X_i^p$

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[d] \\ S/pS \arrow[r, "\varphi"'] & S/pS \end{tikzcd}$$

flat by Thm 1.

∴ ϕ is flat by the local criterion.

→ It suffices to check

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\phi"'] & R/\pi R \arrow[d, "\varphi"] \\ S \arrow[r] & R/\pi^p R \end{tikzcd}$$

is a cocartesian diagram of rings. ⌟

Skipped in the talk.

② Choose $\underline{x} = x_1, \cdots, x_r \subset \mathfrak{m}$ so that

$\overline{\underline{x}} := \underline{x} \bmod \pi R$ is a minimal basis of $\mathfrak{m}/\pi R$.

$\left(r = \operatorname{edim} R/\pi R \right)$

Then $\overline{\underline{x}}$ is Lech independent

in the sense that

$$\left(\overline{\underline{x}}\right) \Big/ \left(\overline{\underline{x}}\right)^2 = \bigoplus_{i=1}^{r} \overline{R} \big/ \left(\overline{\underline{x}}\right) \cdot \overline{x}_i$$

Since $\varphi : R/\pi R \to R/\pi^p R$ is flat,

$\underline{x}^p \bmod \pi^p R$ is again Lech indep.

$\left(\underline{x}^p := x_1^p, \cdots, x_r^p \right)$

$\to$ $\boxed{c := \ell\left(R/(\pi^p, \underline{x}^p) \right) = p^r \cdot \ell\left(R/(\pi^p, \underline{x}) \right)}$ — $(*)$

$\therefore c \geq p^r$.

Let's admit : $\boxed{p^{D+1} \geq c}$ — $(\#)$

where $\boxed{D := \dim R/\pi R.}$

• $c > p^r \underset{(\#)}{\Longrightarrow} r < D+1 \underset{\pi \in N2D(R)}{=} \dim R$

$\Longrightarrow \operatorname{edim} R \le r+1 \le \dim R.$

• $c = p^r \underset{(*)}{\Longrightarrow} \mathfrak{m} = (\pi^p, x)$

$$= (x) + \pi \cdot \pi^{p-1} R$$

$$= (x) + \pi \mathfrak{m}$$

$\underset{\text{NAK}}{\Longrightarrow} \mathfrak{m} = (x)$

$\Longrightarrow \operatorname{edim} R \le r \underset{(\#)}{\le} D+1 = \dim R$

//

Proof of (#)

$$\chi(n) := \ell\left(R/(\pi R + \mathfrak{m}^n)\right) \quad \text{Hilbert-Samuel function}$$

$$= \oslash\, n^D + \cdots$$

By a length computation, one has

$$\chi(n) \le \frac{p}{c}\, \chi(pn + (p-1)r)$$

$$\downarrow$$

$$\oslash \le \frac{p}{c}\, \oslash\, p^D$$

//

Apply Thm 2 to perfectoid towers

$$R = R_0 \to R_1 \to \cdots \to R_i \xrightarrow{t_i} R_{i+1} \to \cdots$$
$$\begin{array}{c} \rotatebox{90}{$\in$} \\ \pi \end{array}$$

① $\pi^p R_1 = pR_1$

②
$$\begin{tikzcd}
R_{i+1}/pR_{i+1} \arrow[r, "\mathrm{Frob}"] \arrow[d, two heads] & R_{i+1}/pR_{i+1} \\
R_{i+1}/\pi R_{i+1} \arrow[ur, "\varphi"] \arrow[r, dashed, "\cong"] & R_i/pR_i \arrow[u, "\overline{t_i}"']
\end{tikzcd}$$

~~Cor 4~~ Thm 4 (H.)

Assume R_i, R_{i+1} are Noeth. local

$\cancel{\pi \in NZD(R_{i+1})}$

Then R_i is regular $\iff$ t_i is flat.

Rem 5 More precisely,

$$\begin{array}{ccc} R_i \text{ is regular} & \iff & t_i \text{ is flat.} \\ \Updownarrow & & \Updownarrow \\ R_i^{s.b} \text{ is regular} & \iff & t_i^{s.b} \text{ is flat.} \end{array}$$

where

$$R^{s.b} = R_0^{s.b} \to R_1^{s.b} \to \cdots \to R_i^{s.b} \xrightarrow{t_i^{s.b}} R_{i+1}^{s.b} \to \cdots$$

is the tilt.