パーフェクトイド塔の傾化対応について

tilt(ing)

§1. Introduction

§2. Perfectoid towers

§3. Tilting correspondences.

§1

What is "tilting correspondence"?

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{"perfectoid objects"} \\ \text{of mixed char.}\end{array}} \arrow[r, squiggly, "\text{tilting}"] \arrow[d] & \boxed{\begin{array}{c}\text{"perfect(oid) objects"} \\ \text{of positive char.}\end{array}} \arrow[d] \\
\text{invariants} \arrow[r, leftrightarrow, "\text{Comparison}"] & \text{invariants}
\end{tikzcd}
$$

$$\downarrow$$

Applications

Perfectoid fields	$K = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} \rightsquigarrow K^\flat = \widehat{\mathbb{F}_p((T))(T^{\frac{1}{p^\infty}})}$ $G_K \cong G_{K^\flat}$
— " — spaces	$X = \mathrm{Spa}(K, \mathcal{O}_K) \rightsquigarrow X^\flat = \mathrm{Spa}(K^\flat, \mathcal{O}_{K^\flat})$ $X_{\acute{e}t} \simeq X^\flat_{\acute{e}t}$ Applications: p-adic Hodge theory
— " — rings	$A = \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]} \rightsquigarrow A^\flat = \widehat{\mathbb{F}_p[[T]][T^{\frac{1}{p^\infty}}]}$ $\mathrm{Perfd}_{/A} \simeq \mathrm{Perfd}_{/A^\flat}$ Direct summand conj. BZM conj.
— " — towers	$\{R_i = \mathbb{Z}_p[p^{\frac{1}{p^i}}]\}_{i\geq 0} \rightsquigarrow \{R_i^{s.\flat} = \mathbb{F}_p[[T]][T^{\frac{1}{p^i}}]\}_{i\geq 0}$ Today Future work.

Toy model of applications of tilting:

<u>Prop 1.1</u> $A = \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$

$$\Rightarrow H^q_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } \forall q > 1.$$

$\because$ We may replace A by $A^\flat = \widehat{\mathbb{F}_p[[T]][T^{\frac{1}{p^\infty}}]}$ because

$$\boxed{H^q_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/p\mathbb{Z}) \simeq H^q_{\text{ét}}(\operatorname{Spec} A^\flat, \mathbb{Z}/p\mathbb{Z})}$$

Then we only have to use the Artin-Schreier seq.

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0 : \text{exact in } \widetilde{(\operatorname{Spec} A)}_{\text{ét}}$$

(small) étale topos //

§2

$\boxed{\text{Fix a prime number } p}$

Def 2.1 (Ishiro-Nakazato-Shimomoto)

R: comm. ring

A perfectoid tower arising from (R, p)

is an inductive system of rings

$$\{R_i\}_{i \geq 0} : R_0 \to R_1 \to \cdots \to R_i \xrightarrow{t_i} R_{i+1} \to \cdots$$

s.t. ⓐ $R_0 = R$

ⓑ $\forall i \geq 0$, $\overline{t_i} : R_i/p \to R_{i+1}/p$ is inj.

ⓒ $\forall i \geq 0$,

$$\begin{tikzcd} R_{i+1}/p \arrow[r, "\mathrm{Frob}"] \arrow[dr, dotted, "\exists (1)\, F_i"'] & R_{i+1}/p \\ & R_i/p \arrow[u, "\overline{t_i}"'] \end{tikzcd}$$

i^{th} Frobenius projection

ⓓ $\forall i \geq 0$, F_i is surj.

ⓔ $\forall i \geq 0$, $p \in \mathrm{rad}(R_i)$

(e.g., R_i is p-adically complete)

ⓕ $\exists\, \varpi \in R_1$ with $\varpi^p R_1 = pR_1$

& $\mathrm{Ker}(F_i) = \varpi \cdot R_{i+1}/pR_{i+1}$ $\forall i \geq 0$

(g) $\forall i \geq 0,\ p \cdot (R_i)_{p\text{-tor}} = (0)$,

where $(R_i)_{p\text{-tor}} := \bigcup_{n>0} (0 :_{R_i} p^n)$

&
$$\begin{tikzcd}
(R_{i+1})_{p\text{-tor}} \arrow[r] \arrow[d, dashed, "\exists(1)"'] \arrow[dr, phantom, "\circlearrowright"] & R_{i+1}/p \arrow[d, "F_i"] \\
(R_i)_{p\text{-tor}} \arrow[r] & R_i/p
\end{tikzcd}$$

Ex 2.2 $R_i := \mathbb{Z}_p[p^{\frac{1}{p^i}}]$

$$\begin{tikzcd}
\mathbb{Z}_p[p^{\frac{1}{p^{i+1}}}]/p \arrow[r, "\mathrm{Frob}"] \arrow[d, two heads] \arrow[dr, dashed] & \mathbb{Z}_p[p^{\frac{1}{p^{i+1}}}]/p \\
\mathbb{Z}_p[p^{\frac{1}{p^{i+1}}}]/p^{\frac{1}{p}} \arrow[r, dashed, "\cong"] & \mathbb{Z}_p[p^{\frac{1}{p^i}}]/p \arrow[u, hook]
\end{tikzcd}$$

Ex 2.3 R: (reduced) $\mathbb{F}_p$-alg.

Then
$$\begin{tikzcd}
R_{i+1} \arrow[r, "\mathrm{Frob}"] \arrow[dr, dashed, "\cong"'] & R_{i+1} \\
& R_i \arrow[u, hook]
\end{tikzcd}
\iff R_{i+1} \cong (R_i)^{\frac{1}{p}}$$

$\therefore \{R_i\}_{i\geq 0} \cong (R \hookrightarrow R^{\frac{1}{p}} \hookrightarrow R^{\frac{1}{p^2}} \hookrightarrow \cdots)$

perfect tower

Def 2.4 $\{R_i\}_{i\geq 0}$: perfd. tower arising from R.

$$R_i^{s.b} \underset{\text{def}}{=} \varprojlim \left(R_i/p \xleftarrow{F_i} R_{i+1}/p \xleftarrow{F_{i+1}} R_{i+2}/p \leftarrow \cdots \right)$$

ith small tilt

$$\longrightarrow \quad \begin{tikzcd} R_{i+1}^{s.b} \arrow[r, "\mathrm{Frob}"] \arrow[dr, "\cong", "\varprojlim_j F_{j+i}"'] & R_{i+1}^{s.b} \\ & R_i^{s.b} \arrow[u, hook, "\varprojlim_j \overline{t_{i+j}}"'] \end{tikzcd}$$

$\therefore$ $\{R_i^{s.b}\}_{i\geq 0}$ is a perfect(oid) tower.

Ex 2.5 $R_i = \mathbb{Z}_p[p^{\frac{1}{p^i}}] \xleftarrow{\cong} \mathbb{Z}_p[T^{\frac{1}{p^i}}]/(T-p)$

$$\begin{tikzcd} R_i = \mathbb{Z}_p[p^{\frac{1}{p^i}}] & \mathbb{Z}_p[T^{\frac{1}{p^i}}]/(T-p) \arrow[l, "\cong"'] \arrow[d] \\ R_i/p & \mathbb{F}_p[T^{\frac{1}{p^i}}]/(T) \arrow[l, "\cong"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{F}_p[T]/(T^{p^{i+1}}) \arrow[r, "\mathrm{Frob}"] \arrow[dr, "\mathrm{can}"'] & \mathbb{F}_p[T]/(T^{p^{i+1}}) \\ & \mathbb{F}_p[T]/(T^{p^i}) \arrow[u, hook, "T \mapsto T^p"'] \end{tikzcd}$$

$\therefore$ $R_i^{s.b} \cong \mathbb{F}_p[[T]]$

$\xleftarrow{\cong} \mathbb{F}_p[[T]]^{\frac{1}{p^i}} = \mathbb{F}_p[[T]][T^{\frac{1}{p^i}}]$

Further constructions:

$\exists$ Frob. lift (Ishizuka, Ishiro-Shimomoto)

affine semigp. rings $\mathbb{Z}_p[\![H]\!]$ $\rightsquigarrow$ $\mathbb{F}_p[\![T]\!][\![H]\!]$

$\mathbb{Z}_p[\![t^2, t^3]\!]$ $\rightsquigarrow$ $\mathbb{F}_p[\![T]\!][\![t^2, t^3]\!]$

log regular $\dfrac{\mathbb{Z}_p[\![x,y,z]\!]}{(xy-pz)}$ $\rightsquigarrow$ $\dfrac{\mathbb{F}_p[\![T,x,y,z]\!]}{(xy-Tz)}$

regular

p-Stanley-Reisner rings $\mathbb{Z}_p\overline{[\![\Delta]\!]}$ $\rightsquigarrow$ $\mathbb{F}_p[\![\Delta]\!]$

$\dfrac{\mathbb{Z}_p[\![x,y]\!]}{(px, py)}$ $\rightsquigarrow$ $\dfrac{\mathbb{F}_p[\![T,x,y]\!]}{(Tx, Ty)}$

- Andreatta-Faltings' const. (H.-Ishiro-Shimomoto)
- (Weakly) étale base change (H.)
- Modification in char. P. (H.)

$$\begin{tikzcd} & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & S \arrow[d] \\ \mathbb{F}_p \arrow[r] & R' \arrow[r] & (S/p)_{red} \end{tikzcd}$$

S admits a perfd tower $\Rightarrow$ so does R.

§3. $\{R_i\}_{i\geq 0}$: a perfd. tower arising from R.

Thm 3.1 (Ishiro-Nakazato-Shimomoto)

$$\mathrm{Ker}(R^{s.b} \xrightarrow{pr_0} R/p) = ({}^{\exists}p^{s.b})$$

Moreover,

$$\begin{tikzcd} ① & R_i^{s.b}/p^{s.b} \arrow[r, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & R_i/p \\ ② & (R_i^{s.b})_{p^{s.b}\text{-tor}} \arrow[u] \arrow[r, dashed, "\cong"] & (R_i)_{p\text{-tor}} \arrow[u] \end{tikzcd}$$

Rem 3.2 $R_i^{s.b}$ is $p^{s.b}$-adically complete.

Prop 3.3 $\forall i \geq 0$

① R_i is local $\Leftrightarrow$ $R_i^{s.b}$ is local

② R_i is Noetherian $\Rightarrow$ $R_i^{s.b}$ is Noetherian

③ $R_i \to R_{i+1}$ is finite $\Rightarrow$ $R_i^{s.b} \to R_{i+1}^{s.b}$ is finite
($\Leftrightarrow$ $R^{s.b}$ is F-finite)

In ② and ③,

"$\Leftarrow$" holds if R_i is p-adically complete.

$\because$ Use Thm 3.1 ① and completeness.

Say R_i is p-torsion free

if $(R_i)_{p\text{-tor}} = 0$

$\Leftrightarrow$ $p \in NZD(R_i)$

$\Big\downarrow$ Thm 3.1 ②,

R_i is p-torsion free $\Leftrightarrow$ $R_i^{s.b}$ is p-torsion free.

$\therefore$ In this case, R_i is CM $\Leftrightarrow$ $R_i^{s.b}$ is CM.

$$\begin{pmatrix}\text{Gor.}\\ \text{CI}\end{pmatrix} \qquad \begin{pmatrix}\text{Gor.}\\ \text{CI}\end{pmatrix}$$

In the general case, the following is useful.

Thm 3.5 (H.)

$\left\{ \widetilde{R_i} := R_i/(R_i)_{p\text{-tor}} \right\}_{i\geq 0}$, $\left\{ (R_i/pR_i)_{red} \right\}_{i \geq 0}$

are perfect(oid) towers.

Moreover, the diagram of rings

$$\begin{array}{ccc} R_i & \longrightarrow & \widetilde{R_i} \\ \downarrow & & \downarrow \\ (R_i/p)_{red} & \longrightarrow & (\widetilde{R_i}/p)_{red} \end{array}$$

is cartesian.

Ex 3.6

$$\begin{tikzcd} \dfrac{\mathbb{Z}_p[[x,y]]}{(px,py)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathbb{Z}_p \arrow[d] \\ \mathbb{F}_p[[x,y]] \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

$$R \xrightarrow{\cong} \tilde{R} \underset{(\tilde{R}/p)_{red}}{\times} (R/p)_{red}$$

$$\rightsquigarrow$$

$$R^{s.b} \xrightarrow{\cong} \widetilde{R^{s.b}} \underset{(\tilde{R}/p)_{red}}{\times} (R/p)_{red}$$

$$\widetilde{R^{s.b}} \cong (\tilde{R})^{s.b}$$

Cor 3.7 R_i : Noeth. local

$\Rightarrow \dim R_i = \dim R_i^{s.b}$

$\because$ $$\dim R_i = \max\{\dim \tilde{R}_i, \dim (R_i/p)_{red}\}$$

$$\|$$

$$\dim R_i^{s.b} = \max\{\dim \widetilde{R_i^{s.b}}, \dim (R_i/p)_{red}\}$$

//

Thm 3.8 (Koszul homology)

$x = p, x_2, \cdots, x_d \subset R_i$

$x^{s.b} = p^{s.b}, x_2^{s.b}, \cdots, x_d^{s.b} \subset R_i^{s.b}$

: identified under Thm 3.1 ①

$\Rightarrow K_\bullet(x; R_i) \cong K_\bullet(x^{s.b}; R_i^{s.b})$ in $D(Ab)$

$\because$ Using exact triangles,

$$K_\bullet(x; R_i) \to K_\bullet(x; \tilde{R}_i) \oplus K_\bullet(x; R_i/p) \to K_\bullet(x; \tilde{R}_i/p) \dashrightarrow$$

$$K_\bullet(x^{s.b}; R_i^{s.b}) \to K_\bullet(x^{s.b}; R_i^{s.b}) \oplus K_\bullet(x; R_i/p) \to K_\bullet(x; \tilde{R}_i/p) \dashrightarrow$$

one can reduce to the p-torsion free case.

Then
$$\begin{array}{ccc} K_\bullet(x; R) & \xrightarrow{\cong} & K_\bullet(x'; R/p) \\ & & \uparrow \cong \\ K_\bullet(x^{s.b}; R^{s.b}) & \xrightarrow{\cong} & K_\bullet(x^{s.b\prime}; R^{s.b}/p^{s.b}) \end{array}$$
//

Cor 3.9 R_i : Noeth. local.

$\Rightarrow \operatorname{depth} R_i = \operatorname{depth} R_i^{s.b}$

$\therefore R_i$ is CM $\Leftrightarrow R_i^{s.b}$ is CM.

$\because$ By "depth sensitivity" //

Ex 3.10 Δ : a simplicial cpx

$\to$ Stanley-Reisner ring

$$\mathbb{Z}_p[\Delta] = \mathbb{Z}_p[x_1, \cdots, x_n] \Big/ \begin{pmatrix} \text{squarefree} \\ \text{monomials} \end{pmatrix}$$

Then

$$\mathbb{Z}_p\overline{[\Delta]} := \mathbb{Z}_p[\Delta] \big/ (x_1 - p) \quad \text{is CM}$$

$$\Leftrightarrow \mathbb{F}_p[\Delta] \text{ is CM.}$$

$\uparrow$

One can apply Reisner's criterion.

Thm 3.11 R_i, R_{i+1} : Noeth. local

TFAE

① R_i is regular.

①$^{s.b}$ $R_i^{s.b}$ is regular.

② $R_i \to R_{i+1}$ is flat.

②$^{s.b}$ $R_i^{s.b} \to R_{i+1}^{s.b}$ is flat.

∵ ② $\Leftrightarrow$ ②$^{s.b}$:

Use Thm 3.5 to reduce to the p-torf. case, and apply the local criterion of flatness.

①$^{s.b}$ $\Leftrightarrow$ ②$^{s.b}$: By Kunz's thm.

① $\Leftrightarrow$ ② : By ② $\Leftrightarrow$ ①$^{s.b}$, WMA R_{i+1} : p-torf.

Then we are reduced to prove the following. //

Thm 3.12 (Gabber-Lurie)

$(R, \mathfrak{m})$: Noeth. local with $p \in \mathfrak{m}$

${}^{\exists}\pi \in NZD(R)$ s.t. $p \in \pi^p R$.

Then R is regular $\Leftrightarrow$ $\varphi : R/\pi \to R/\pi^p$ is flat.

$$\begin{array}{ccc} R/\pi & \to & R/\pi^p \\ \cup & & \cup \\ x & \mapsto & x^p \end{array}$$

Thm 3.13 (Étale cohomology)

Assume R: p-adically henselian

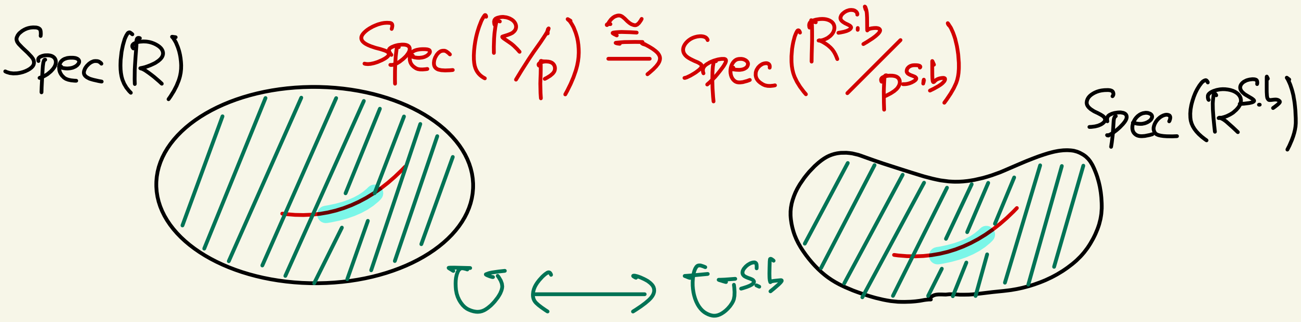
G: torsion abel. gp.

$\mathrm{Spec}(R) \setminus V(p) \subset U \underset{\text{open}}{\subset} \mathrm{Spec}(R)$

$\updownarrow$

$\mathrm{Spec}(R^{s.b}) \setminus V(p^{s.b}) \subset U^{s.b} \underset{\text{open}}{\subset} \mathrm{Spec}(R^{s.b})$

$\Rightarrow R\Gamma_{\acute{e}t}(U, G) \cong R\Gamma_{\acute{e}t}(U^{s.b}, G)$



$\because$ Following Česnavičius-Scholze's proof

in the case of perfectoid <ins>rings</ins>,

use arc (hyper)descent results

of Bhatt-Mathew

to reduce to the case of $\prod_{i \in I} V_i$,

V_i: perfectoid <ins>valuation</ins> rings. //