パーフェクトイド塔の標準分解とその応用

$\boxed{\text{Fix a prime number } p}$

perfectoid theory

$$\boxed{\text{mixed char.}} \longleftrightarrow \boxed{\text{positive char.}}$$

perfectoid rings

$$\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]} \longleftarrow \quad \longrightarrow \widehat{\mathbb{F}_p[[T]][T^{\frac{1}{p^\infty}}]}$$

$$\begin{array}{ccc} \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]} & & \widehat{\mathbb{F}_p[[T]][T^{\frac{1}{p^\infty}}]} \\ \cup & & \cup \\ \vdots & & \vdots \\ \cup & & \cup \\ \mathbb{Z}_p[p^{\frac{1}{p^2}}] & & \mathbb{F}_p[[T^{\frac{1}{p^2}}]] \\ \cup & & \cup \\ \mathbb{Z}_p[p^{\frac{1}{p}}] & \text{perfectoid towers} & \mathbb{F}_p[[T^{\frac{1}{p}}]] \\ \cup & & \cup \\ \mathbb{Z}_p & & \mathbb{F}_p[[T]] \end{array}$$

§1. Backgrounds for perfectoid rings

§2. Results for perfectoid towers.

§ 1.

- A : a perfectoid ring

$\overset{}{\underset{\text{def}}{\iff}}$ ${}^{\exists}\varpi \in A$ with $p \in \varpi^p A$ s.t.

① A is ϖ-adically complete

②
$$\begin{tikzcd} A/\varpi^p \arrow[r, "\mathrm{Frob}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A/\varpi^p \\ A/\varpi \arrow[ur, dotted, "\cong"'] & \end{tikzcd}$$

③
$$\begin{array}{ccc} A_{\varpi\text{-tor}} & \xrightarrow{\cong} & A_{\varpi\text{-tor}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & x^p \end{array}$$

- $A^\flat \underset{\text{def}}{=} \varprojlim \left(A/p \xleftarrow{\mathrm{Frob}} A/p \xleftarrow{\mathrm{Frob}} \cdots \right)$: tilt of A

Prop 1.1 ① $B := A/A_{\varpi\text{-tor}}$ is perfectoid.

② $(A/\varpi)_{red}$ is perfect(oid).

③
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ (A/\varpi)_{red} \arrow[r] & (B/\varpi)_{red} \end{tikzcd}$$

is a cartesian diagram of rings.

Cor 1.2 A is reduced.

§ 2.

- $\{R_i\}_{i\geq 0} = (R = R_0 \to R_1 \to R_2 \to \cdots)$

  : a perfectoid tower arising from (R, p).

$\overset{\text{def}}{\iff}$ $\exists\, \varpi \in R_1$ with $pR_1 = \varpi^p R_1$ s.t. $\forall i \geq 0$,

① R_i is p-adically complete

②
$$\begin{tikzcd} R_{i+1}/p \arrow[r, "\mathrm{Frob}"] \arrow[d] \arrow[dr, "F_i"] & R_{i+1}/p \\ R_{i+1}/\varpi \arrow[r, "\cong"] & R_i/p \arrow[u, hook] \end{tikzcd}$$

③ $\varpi \cdot (R_i)_{p\text{-tor}} = (0)$

& F_i induces a bij.

$$(R_{i+1})_{\varpi\text{-tor}} \xrightarrow{\cong} (R_i)_{p\text{-tor}}.$$

- $R_i^{s.b} \underset{\text{def}}{=\!=} \varprojlim \left(R_i/p \xleftarrow{F_i} R_{i+1}/p \xleftarrow{F_{i+1}} \cdots \right)$

  : i^{th} small tilt.

Ex 2.1 ① $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[p^{\frac{1}{p}}] \hookrightarrow \mathbb{Z}_p[p^{\frac{1}{p^2}}] \hookrightarrow \cdots$

$$\rightsquigarrow$$

$$\mathbb{F}_p[[T]] \hookrightarrow \mathbb{F}_p[[T^{\frac{1}{p}}]] \hookrightarrow \mathbb{F}_p[[T^{\frac{1}{p^2}}]] \hookrightarrow \cdots$$

A similar construction works for complete local (log-) regular rings.

② $A \xrightarrow{id} A \xrightarrow{id} A \xrightarrow{id} \cdots$ (A: perfectoid ring)

$$\rightsquigarrow$$

$$A^\flat \xrightarrow{id} A^\flat \xrightarrow{id} A^\flat \xrightarrow{id} \cdots$$

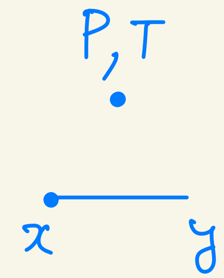
③ $\exists$ a construction from Frobenius lifts (Ishizuka, Ishiro-Shimomoto)

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ A/p \arrow[r, "\mathrm{Frob}"'] & A/p \end{tikzcd}$$

e.g., $R = \mathbb{Z}_p[[x,y]] / (px, py)$

$$\rightsquigarrow$$

$$R^{s.\flat} = \mathbb{F}_p[[T, x, y]] / (Tx, Ty)$$



More generally, for a simplicial cpx. Δ,

$$R = \mathbb{Z}_p\overline{[[\Delta]]} = \frac{\mathbb{Z}_p[[x_2, \cdots, x_d]]}{\left(\begin{array}{c}\text{squarefree monomials} \\ \text{w.r.t. } p, x_2, \cdots, x_d\end{array}\right)} : \text{p-SR ring (O. Strahan)}$$

$$\rightsquigarrow$$

$R^{s.\flat} = \mathbb{F}_p[[\Delta]]$: Stanley-Reisner ring

Thm 2.2 (H.)

① $\left\{ S_i := R_i / (R_i)_{\varpi\text{-tor}} \right\}_{i\geq 0}$ is a perfectoid tower.

② $\left\{ (R_i/p)_{red} \right\}_{i\geq 0}$ is a perfect(oid) tower.

i.e., $\cong \left[\overline{R} := (R/p)_{red} \hookrightarrow (\overline{R})^{\frac{1}{p}} \hookrightarrow \cdots \right]$

③
$$\begin{tikzcd} R_i \arrow[r] \arrow[d] & S_i \arrow[d] \\ (R_i/p)_{red} \arrow[r] & (S_i/p)_{red} \end{tikzcd}$$

is a cartesian diagram of rings ($\forall i \geq 0$).

Cor 2.3 R_i is reduced for $\forall i \geq 0$.

Ex 2.4
$$\begin{tikzcd} \dfrac{\mathbb{Z}_p[[x,y]]}{(px,py)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathbb{Z}_p \arrow[d] \\ \mathbb{F}_p[[x,y]] \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

Fact 2.5 $\exists p^{s.b} \in R^{s.b}$ s.t. $R^{s.b}/p^{s.b} \xrightarrow{\cong} R/p$

Moreover, $p^{s.b} \in NZD(R^{s.b}) \Longleftrightarrow p \in NZD(R)$.

Cor 2.6 Assume R is a Noetherian local ring ($\Leftrightarrow$ so is $R^{s.b}$)

① $\dim R = \dim R^{s.b}$

② $\underline{x} = p, x_2, \cdots, x_n \subset R$

$\Updownarrow$

$\underline{x}^{s.b} = p^{s.b}, x_2^{s.b}, \cdots, x_n^{s.b} \subset R^{s.b}$

$\Rightarrow K_\bullet(\underline{x}; R) \cong K_\bullet(\underline{x}^{s.b}; R^{s.b})$ in $D(Ab)$

③ $\operatorname{depth} R = \operatorname{depth} R^{s.b}$

④ R is CM $\Leftrightarrow R^{s.b}$ is CM.

Cor 2.7 Δ : simplicial cpx.

TFAE

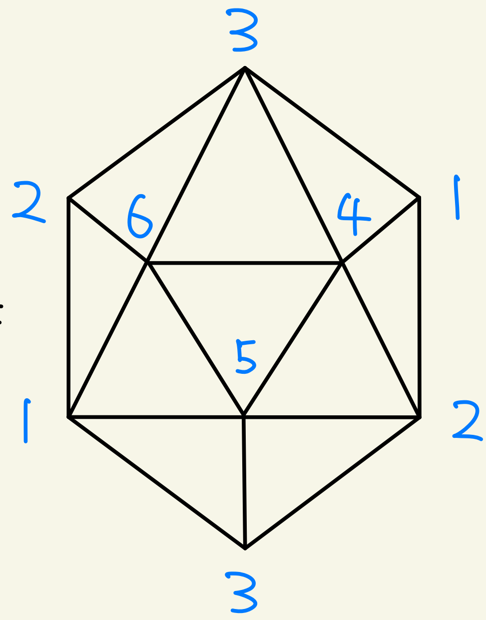
① $\mathbb{Z}_p\overline{[\Delta]}$ is CM

② $\mathbb{F}_p[\Delta]$ is CM.

③ $\widetilde{H}_q(\operatorname{lk} F; \mathbb{F}_p) = 0$ for $^\forall F \in \Delta$ $^\forall q < \dim \operatorname{lk} F$

(Reisner's criterion)

Ex 2.8

$\Delta := \mathbb{R}\mathbb{P}^2 =$ 

Then

$$\tilde{H}_1(\mathbb{R}\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z}$$

and $\mathbb{F}_p[\Delta]$ is $\begin{cases} \text{not CM} & (p = 2) \\ \text{CM} & (p \neq 2) \end{cases}$

$\therefore$ $\mathbb{Z}_p\overline{[\Delta]}$ is ________ //