パーフェクトイド塔の被約性について

$\boxed{\text{Fix a prime number } p}$

§1. Backgrounds

A: perfectoid ring

Prop 1.1 A is reduced.

$\because$ Case 1 : $A \supset \mathbb{F}_p$.

Then A is perfectoid

$$\iff A \text{ is perfect} \overset{\text{def}}{\iff} \varphi : \begin{array}{ccc} A & \to & A \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \mapsto & x^p \end{array} \text{ is bij.}$$

$$\implies A \text{ is reduced} \iff \varphi \text{ is inj.}$$

Case 2 : A is p-torsion free $\iff p \in NZD(A)$

Then A is perfectoid

$\iff \exists\, \varpi \in A$ with $p \in \varpi^p A$

ⓐ A is ϖ-adically complete.

ⓑ $$\begin{array}{ccc} A/\varpi A & \xrightarrow{\cong} & A/\varpi^p A \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & x^p \end{array}$$

$\forall x \in A$

$$x^p = 0 \underset{ⓑ}{\Rightarrow} \overset{\overset{0}{\parallel}}{x^p} = \underbrace{\varpi^p}_{NZD} x_1^p$$

$$\Rightarrow x_1^p = 0 \underset{ⓑ}{\Rightarrow} x_1 = \varpi x_2$$

$$\vdots$$

$$\Rightarrow x \in \bigcap_{n>0} \varpi^n A \underset{ⓐ}{=} (0).$$

Case 3: The general case

Reduce to Case 1 & 2 by the following

Prop 1.2 ① $B := A/A_{p\text{-tor}}$ is perfectoid.

② $(A/pA)_{red}$ is a perfect $\mathbb{F}_p$-alg.

③

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ (A/pA)_{red} \arrow[r] & (B/pB)_{red} \end{tikzcd}$$

is a cartesian diagram of rings.

§2. Main results

$\{R_i\}_{i\geq 0} = (R_0 \to R_1 \to \cdots)$: a perfectoid tower arising from (R, I_0).

Thm 2.1 (H.)

① $\left\{ S_i := R_i / (R_i)_{I_0\text{-tor}} \right\}_{i\geq 0}$ is a perfectoid tower.

② $\left\{ \left(R_i / I_0 R_i \right)_{red} \right\}_{i\geq 0}$ is a perfect tower.

③
$$\begin{tikzcd}
R_i \arrow[r] \arrow[d] & S_i \arrow[d] \\
\left(R_i / I_0 R_i \right)_{red} \arrow[r] & \left(S_i / I_0 S_i \right)_{red}
\end{tikzcd}$$
is a cartesian diagram of rings.

Cor 2.2 Suppose $\forall i \geq 0$, R_i : I_0-adically separated.
(eg., R_i : Noetherian)

Then $\forall i \geq 0$

① $R_i \to R_{i+1}$ is inj.

② R_i is reduced.

Cor 2.3 Fix $i \geq 0$,

and assume R_i : Noetherian local.

① $\dim R_i = \dim R_i^{s.b}$

② $\underline{x} = x_1, \cdots, x_n \subset R_i$ s.t. $I_0 R_i = (x_1)$

$\Updownarrow$

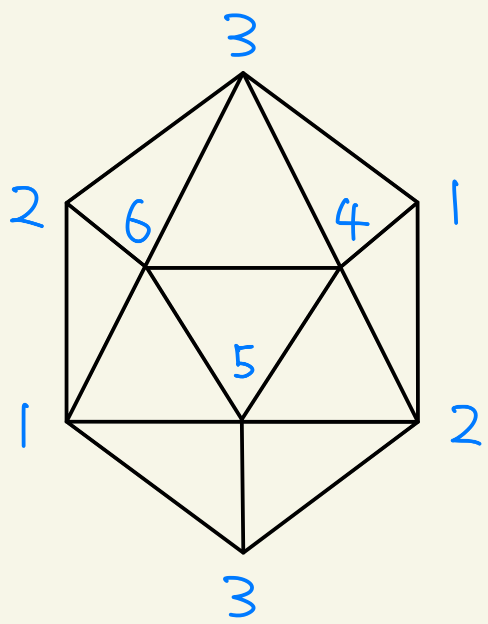
$\underline{x}^{s.b} = x_1^{s.b}, \cdots, x_n^{s.b} \subset R_i^{s.b}$

$\Rightarrow K_\bullet(\underline{x}; R_i) \cong K_\bullet(\underline{x}^{s.b}; R_i^{s.b})$ in $D(Ab)$.

③ $\operatorname{depth} R_i = \operatorname{depth} R_i^{s.b}$

④ R_i is Cohen-Macaulay $\Leftrightarrow$ so is $R_i^{s.b}$

Ex 2.4 (Stanley-Reisner rings)

Δ: 

The standard minimal triangulation of $\mathbb{R}P^2$.

$$R = \frac{\mathbb{Z}_p[[x_2, x_3, x_4, x_5, x_6]]}{\left(\begin{array}{ll} p\, x_2x_3, & x_2x_3x_4, \\ p\, x_2x_5, & x_2x_4x_6, \\ p\, x_2x_6, & x_2x_5x_6, \\ p\, x_4x_5, & x_3x_4x_5, \\ p\, x_4x_6, & x_3x_5x_6 \end{array}\right)}$$

$\rightarrow$ R admits a perfectoid tower

(Ishiro-Shimomoto '25)

$$R^{s.b} = \mathbb{F}_p[[\Delta]] = \frac{\mathbb{F}_p[[x_1, x_2, x_3, x_4, x_5, x_6]]}{\left(\begin{array}{ll} x_1x_2x_3, & x_2x_3x_4, \\ x_1x_2x_5, & x_2x_4x_6, \\ x_1x_2x_6, & x_2x_5x_6, \\ x_1x_4x_5, & x_3x_4x_5, \\ x_1x_4x_6, & x_3x_5x_6 \end{array}\right)}$$

By Reisner's criterion,

$$R^{s.b} \text{ is } \begin{cases} \text{Cohen-Macaulay} & (p \neq 2) \\ \text{not } \text{——〃——} & (p = 2) \end{cases}$$

$\therefore$ R is ——————〃——————