分岐理論から生じるパーフェクトイド塔について

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$\boxed{\text{Perfectoid rings/spaces}}$

Arith. geom.

Comm. alg. in mixed char.

Sing. theory in mixed char.

generalization /refinement

$\boxed{\text{Perfectoid towers}}$

§1. Perfectoid rings/towers.

§2. Our results.

$\boxed{\text{Fix a prime number } p.}$

Ring = commutative with 1.

§1.

Def 1.1 (Bhatt-Morrow-Scholze '19)

A : p-torsion free ring

A is perfectoid

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists\, \varpi \in A$ with $p \in \varpi^p A$ s.t.

① A is ϖ-adically complete

② $$\begin{array}{ccc} A/\varpi A & \xrightarrow{\cong} & A/\varpi^p A \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & x^p \end{array}$$

Ex 1.2 ① $\mathcal{O}_K$ is perfectoid,

where $K = \mathbb{C}_p, \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$

② $R_\infty := \bigcup_{n \geq 0} R_n$, $R_n := \mathbb{Z}_p[p^{\frac{1}{p^n}}]$

$\Rightarrow \widehat{R_\infty}^p$ is perfectoid

Obs 1.3

$$
\begin{tikzcd}
R_{n+1}/p \arrow[r, "\mathrm{Frob} \,=:\, \varphi"] \arrow[d] \arrow[dr, "\exists"', color=blue] & R_{n+1}/p \\
R_{n+1}/p^{\frac{1}{p}} \arrow[r, "\cong", "\exists"', color=teal] & R_n/p \arrow[u, hook]
\end{tikzcd}
$$

Def 1.4 (Ishiro-Nakazato-Shimomoto '25)

$\{R_n\}_{n\geq 0} = (R_0 \to R_1 \to R_2 \to \cdots)$: inductive system of p-torsion free rings.

$R_0 \supset I_0 \ni p$, I_0 : principal ideal

$\{R_n\}_{n\geq 0}$ is a perfectoid tower arising from (R_0, I_0)

$\overset{\text{def}}{\Longleftrightarrow}$ ① R_n is I_0-adically Zariskian $\forall n \geq 0$

i.e., $I_0 R_n \subset {}^{\forall}\underset{\text{maximal}}{\mathfrak{m}} \subset R_i$

② ${}^{\exists} I_1 \subset R_1$: principal ideal s.t. $I_1^p = I_0 R_1$

and

$$
\begin{tikzcd}
R_{n+1}/I_0 R_{n+1} \arrow[r, "\varphi"] \arrow[d] \arrow[dr, "\exists"', color=blue] & R_{n+1}/I_0 R_{n+1} \\
R_{n+1}/I_1 R_{n+1} \arrow[r, "\cong", "\exists"', color=teal] & R_n/I_0 R_n \arrow[u, hook]
\end{tikzcd}
\qquad ({}^{\forall} n \geq 0)
$$

Prop 1.5 $\widehat{R_\infty} := \widehat{\varinjlim_{n\geq 0} R_n}$ is a perfectoid ring.

Ex 1.6 ① A: perfectoid ring

$\Rightarrow$ $A \xrightarrow{id} A \xrightarrow{id} A \xrightarrow{id} \cdots$

is a perfectoid tower arising from $(A, \mathcal{O}\text{-}P)$.

② A complete regular local ring of residue char. p admits a perfectoid tower.

e.g., $R := W(k)[[x_2, \cdots, x_d]]$ (k: perfect field of char. p)

$\Rightarrow R \hookrightarrow R[p^{\frac{1}{p}}, x_2^{\frac{1}{p}}, \cdots, x_d^{\frac{1}{p}}] \hookrightarrow R[p^{\frac{1}{p^2}}, x_2^{\frac{1}{p^2}}, \cdots, x_d^{\frac{1}{p^2}}] \hookrightarrow \cdots$

is a perfectoid tower arising from (R, p).

Problem How does one construct perfectoid towers?

① (Ishizuka '26)

$\exists$ a construction from certain "prisms."

② (H. '26) Fiber products,

Étale base change

Today: a construction from ramification theory

§2. We work in the setting of Faltings:

$R := W(k)[[x_2, \cdots, x_d]]$

$R_n := W(k)[p^{\frac{1}{p^n}}, x_2^{\frac{1}{p^n}}, \cdots, x_d^{\frac{1}{p^n}}]$ $(n \geq 0)$

$R \hookrightarrow S$: finite ext. of normal local domains s.t. $R[\frac{1}{p}] \hookrightarrow S[\frac{1}{p}]$ is étale.

$S_n :=$ the int. closure of R_n in $(R_n \otimes S)[\frac{1}{p}]$

Thm 2.1 (Faltings '88)

$\mathfrak{P}_n := (p^{\frac{1}{p^n}}) \in \operatorname{Spec} R_n$: lying over p

$\mathfrak{d}_{S_{n,\mathfrak{P}_n}/R_{n,\mathfrak{P}_n}} = (p^{\delta_n})$ $\exists\, \delta_n \in \mathbb{Q}_{>0}$

discriminant

① $\lim_{n \to \infty} \delta_n = 0$.

② $p^{\delta_n} \cdot \operatorname{Cok}(R_{n+1} \otimes_{R_n} S_n \to S_{n+1}) = 0$.

Thm 2.2 (Andreatta '06)

$\exists\, \varepsilon \in (0,1) \cap \mathbb{Q}$

$\exists\, N \geq 0$ s.t. ① $p^{\varepsilon} \in R_N$

② $\forall n \geq N$,

$$\begin{tikzcd} S_{n+1}/p^{\varepsilon} \arrow[r, "\varphi"] \arrow[dr, "\exists"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & S_{n+1}/p^{\varepsilon} \\ & S_n/p^{\varepsilon} \arrow[u, hook] \end{tikzcd}$$

Thm 2.3 (H.-Ishiro-Shimomoto '26)

After replacing N by a larger one,

$\{S_n\}_{n \geq N}$ is a perfectoid tower arising from (S_N, p^{ε}).

Rem 2.4

Faltings proved ① $\widehat{S_\infty}$: perfectoid

② $R_\infty \to S_\infty$ is almost étale

Scholze, Kedlaya-Liu generalized to

every perfectoid rings (The almost purity theorem)

Future work

- Formulate ② for $\{S_n\}_{n \geq N}$.
- Generalize ① ② to every perfectoid towers.