パーフェクトイド塔の理論とその応用について

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§1. Perfectoid towers

§2. Applications

(§3. My recent works)

$\boxed{\text{Fix a prime number } p.}$

Perfectoid theory (Scholze '12):

$$\boxed{\text{mixed char. } (0,p)} \overset{\text{tilting}}{\rightsquigarrow} \boxed{\text{char. } p}$$

$$\underline{\mathbb{Z}_p \subset \mathbb{Z}_p[p^{\frac{1}{p}}] \subset \cdots \subset \mathbb{Z}_p[p^{\frac{1}{p^i}}] \subset} \cdots \subset \underline{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]^\wedge}$$

perfectoid towers　　perfectoid rings

$$\underline{\mathbb{F}_p[[T]] \subset \mathbb{F}_p[[T^{\frac{1}{p}}]] \subset \cdots \subset \mathbb{F}_p[[T^{\frac{1}{p^i}}]] \subset} \cdots \subset \underline{\mathbb{F}_p[[T^{\frac{1}{p^\infty}}]]}$$

§1. Perfectoid towers

Def ([INS25, Thm 3.50])

A ring A is perfectoid

$\underset{\text{def}}{\iff}$ ${}^{\exists}\varpi \in A$ s.t.

① $p \in \varpi^p A$ and A is ϖ-adically complete

② Frob. on $A/\varpi^p A$ induces $A/\varpi A \xrightarrow{\cong} A/\varpi^p A$

③ $\underbrace{A[\varpi^\infty]}_{= \{x \in A \mid {}^{\exists} n > 0 \text{ s.t. } \varpi^n x = 0\}} \to A[\varpi^\infty]\ ;\ x \mapsto x^p$ is bij.

Ex An $\mathbb{F}_p$-alg. A is perfectoid

$\iff$ A is perfect (i.e., $\varphi : A \to A\ ;\ x \mapsto x^p$ is bij.)

Obs R : $\mathbb{F}_p$-alg., domain

(reduced, more generally)

$$R^{\frac{1}{p^i}} := \{ x \in \overline{\mathrm{Frac}(R)} \mid x^{p^i} \in R \} \quad ({}^{\forall} i \geq 0)$$

① $R_\infty := \bigcup_{i \geq 0} R^{\frac{1}{p^i}}$ is perfect(oid).

②
$$\begin{tikzcd}
R^{\frac{1}{p^{i+1}}} \arrow[r, "\varphi"] \arrow[dr, dotted, "\cong"'] & R^{\frac{1}{p^{i+1}}} \\
 & R^{\frac{1}{p^i}} \arrow[u, hook] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

Def (Ishiro-Nakazato-Shimomoto [INS25, Def 3.21])

A perfectoid tower is

- $\{R_i\}_{i\geq 0} = (R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots)$: an inductive system of rings
- $I_0 \subset R_0$: ideal.

s.t. (a) $p \in I_0$.

(b) (c) (d) $\forall i \geq 0$

$$\begin{tikzcd} R_{i+1}/I_0R_{i+1} \arrow[r, "\varphi"] \arrow[dr, two heads, "\exists F_i"'] & R_{i+1}/I_0R_{i+1} \\ & R_i/I_0R_i \arrow[u, hook, "\overline{t_i}"'] \end{tikzcd}$$

(e) $\forall i \geq 0$, R_i is I_0-adically complete. (for simplicity)

(f) • I_0 : principal, and

• $\exists I_1 \subset R_1$: principal ideal s.t.

$I_1^p = I_0R_1$ and $\forall i \geq 0$, F_i induces

$$R_{i+1}/I_1R_{i+1} \xrightarrow{\cong} R_i/I_0R_i.$$

(g) $\forall i \geq 0$, $R_i[I_0^\infty] = R_i[I_0]$ and

$$\begin{tikzcd} R_{i+1}[I_0] \arrow[r, hook] \arrow[d, dashed, "\exists"', "\cong"] & R_{i+1} \arrow[r] & R_{i+1}/I_0R_{i+1} \arrow[d, "F_i"] \\ R_i[I_0] \arrow[r, hook] & R_i \arrow[r] & R_i/I_0R_i \end{tikzcd}$$

$\underline{\text{Ex}}$ ① $\{R_i\}_{i\geq 0}$ is a perfectoid tower with $I_0 = (0)$

$\Longleftrightarrow$ $\{R_i\}_{i\geq 0} \cong \{R^{\frac{1}{p^i}}\}_{i\geq 0}$ for $^{\exists}R$: reduced $\mathbb{F}_p$-alg.

perfect tower

② $\mathbb{Z}_p \subset \mathbb{Z}_p[p^{\frac{1}{p}}] \subset \mathbb{Z}_p[p^{\frac{1}{p^2}}] \subset \cdots$

$$\begin{array}{cc} \cup & \cup \\ (p) & (p^{\frac{1}{p}}) \end{array}$$ is a perfectoid tower.

A similar construction also works for:

- complete regular local rings
  (e.g., $W(k)[[x_2, \cdots, x_d]]$, k: perfect field $\supset \mathbb{F}_p$)
- complete local log-regular rings
  (e.g., $W(k)[[x,y,z]]/(xy - pz)$)

They are included by the construction of Ishizuka.

($A \circlearrowleft \phi$ ← Frobenius lift) [Ish24]

In what follows,

$$\boxed{(\{R_i\}_{i\geq 0}, I_0) : \text{perfectoid tower}}$$

Def

① $R_i^{s.b} \underset{\text{def}}{=} \varprojlim \left(R_i/I_0R_i \xleftarrow{F_i} R_{i+1}/I_0R_{i+1} \xleftarrow{F_{i+1}} \cdots \right)$

: i th small tilt

② $I_0^{s.b} \underset{\text{def}}{=} \operatorname{Ker}\left(R_0^{s.b} \xrightarrow{\text{pr}_0} R_0/I_0\right)$

Thm ([INS25, Thm 3.35 (2), Lem 3.39, Prop 3.42 (2)])

①
$$\begin{tikzcd} R_i^{s.b}[I_0^{s.b}] \arrow[r] \arrow[d, dashed, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & R_i^{s.b}/I_0^{s.b}R_i^{s.b} \arrow[d, "\cong"] \\ R_i[I_0] \arrow[r] & R_i/I_0R_i \end{tikzcd}$$

② $(\{R_i^{s.b}\}_{i\geq 0}, I_0^{s.b})$ is a perfectoid tower.

③ R_i : Noetherian $\Longleftrightarrow$ $R_i^{s.b}$: Noetherian

§2. Applications

① (Ishiro-Nakazato-Shimomoto '25)

A finiteness thm. on the divisor class group of local log-regular rings.

Thm ([INS25, Thm 4.13])

R: local log-regular ring of mixed char. $(0,p)$

$\mathrm{Cl}(R)$: the divisor class gp.

If $\widehat{R^{sh}}$ is locally factorial,

(R^{sh}: the strict henselization)

then $\# \mathrm{Cl}(R)[\ell^n] < \infty$ for $\forall \ell \neq p$ $\forall n > 0$.

② (Ishiro-Shimomoto '25)

A connection with lim Cohen-Macaulay sequences

(Serre's positivity conj. on multiplicities.)

Thm ([IS25, Thm 3.19])

R_i: Noeth. local domain of mixed char. $(0,p)$ $\forall i \geq 0$.

$\Rightarrow$ $\{R_i\}_{i\geq 0}$: lim Cohen-Macaulay seq. $/R$.

③ (Ishiro-Shimomoto '25)

A thm. on the deformation of perfectoid purity.

(To be updated in [IS25])

<u>Conj</u> $(R, \mathfrak{m})$: complete Gorenstein local ring of mixed char. $(0, p)$.

R/pR : F-pure domain $\Rightarrow$ R : perfectoid pure ?

<u>Thm</u> <u>Conj</u> is true if :

- $R/\mathfrak{m}$: F-finite
- $\exists$ $R = R_0 \to R_1 \to \cdots$: a perfectoid tower

$$\begin{array}{l} R_0 \\ \cup \\ I_0 \end{array}$$

s.t. R is I_0-torsion free.

(e.g., R is (Gor.) log-regular.)

References

[INS25] S. Ishiro, K. Nakazato, K. Shimomoto, *Perfectoid towers and their tilts: with an application to the étale cohomology groups of local log-regular rings*, Algebra Number Theory **19**(12), (2025), 2307–2358.

[Ish24] R. Ishizuka, *Perfectoid towers generated from prisms*, preprint, (2024). arXiv:2409.15785.

[IS25] S. Ishiro, K. Shimomoto, *δ-rings, perfectoid towers, and lim Cohen-Macaulay sequences*, preprint, (2025). arXiv:2509.06527.

[H26] K. Hayashi, *Structural properties of perfectoid towers*, in preparation.

[HIS26] K. Hayashi, S. Ishiro, K. Shimomoto, *An application of Fontaine's monoidal maps to perfectoid towers*, in preparation.

§3. My recent works.

Thm

$$\left.\begin{array}{l} S_i := R_i / R_i[I_0] \\ \overline{R_i} := \left(R_i / I_0 R_i\right)_{red} \\ \overline{S_i} := \left(S_i / I_0 S_i\right)_{red} \end{array}\right\} \text{form perfectoid towers.}$$

Moreover,
$$\begin{array}{ccc} R_i & \longrightarrow & S_i \\ \downarrow & & \downarrow \\ \overline{R_i} & \longrightarrow & \overline{S_i} \end{array}$$
is a cartesian diagram of rings.

Cor ① $t_i : R_i \to R_{i+1}$ is inj.

② R_i is reduced.

③ $\underline{x} = x_1, \cdots, x_n \subset R_0$, $I_0 = (x_1)$

$\Updownarrow$

$\underline{x}^{s.b} = x_1^{s.b}, \cdots, x_n^{s.b} \subset R_0^{s.b}$

$\Rightarrow K(\underline{x}; R_i) \cong K(\underline{x}^{s.b}; R_i^{s.b})$ in $D(Ab)$.

④ R_i : Noetherian local

$$\Rightarrow \begin{cases} \dim R_i = \dim R_i^{s.b} \\ \operatorname{depth} R_i = \operatorname{depth} R_i^{s.b} \end{cases}$$

$\therefore$ R_i is Cohen-Macaulay $\Leftrightarrow$ so is $R_i^{s.b}$.