A refinement of Fontaine's monoidal map for perfectoid towers and its applications

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$\boxed{\text{Fix a prime number } p.}$

$$\begin{tikzcd}
R_0 \arrow[r] & R_1 \arrow[r] & \cdots \arrow[r] & \widehat{R_\infty} \\
R_0^{s.b} \arrow[r] \arrow[u, dotted] & R_1^{s.b} \arrow[r] \arrow[u, dotted] & \cdots \arrow[r] & R_i^{s.b} \arrow[r] \arrow[ul, dotted] & \cdots \arrow[r] & \widehat{R_\infty}^{\flat} \arrow[u, "\#"]
\end{tikzcd}$$

§2 §1

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§1. A: perfectoid ring $\not\supset \mathbb{F}_p$,

$\rightarrow$ $\exists \varpi \in A$ with $p \in \varpi^p A$.

s.t. ① A is ϖ-adically complete

($\Leftrightarrow$ p-adically complete)

② $\exists\, \varpi^{\frac{1}{p}}, \varpi^{\frac{1}{p^2}}, \cdots \in A$ $\quad$ ($\varpi =$ "$p^{\frac{1}{p}}$")

Def 1.1 (Tilts)

$$A^\flat \underset{\text{def}}{=} \varprojlim \left(A/pA \xleftarrow{\text{Frob}} A/pA \xleftarrow{\text{Frob}} \cdots \right)$$

This is a perfect $\mathbb{F}_p$-alg.

Prop-Def 1.2 (Fontaine ~'83)

$\varprojlim \left(A \xleftarrow{x \mapsto x^p} A \xleftarrow{x \mapsto x^p} \cdots\right) \longrightarrow A^\flat$

is an isom. of monoids.

$$\#: A^\flat \xleftarrow{\cong} \varprojlim_{x \mapsto x^p} A \xrightarrow{\mathrm{pr}_0} A \quad \text{Fontaine's monoidal map}$$

$$\begin{array}{ccc} \cup\!\!\!\cup & & \cup\!\!\!\cup \\ \varpi^\flat := (\varpi, \varpi^{\frac{1}{p}}, \varpi^{\frac{1}{p^2}}, \dots) & \mapsto & \varpi \end{array}$$

Thm 1.3 (Eto-Horiuchi-Shimomoto '24)

Assume A is ϖ-torsion free.

($\Leftrightarrow$ $A^\flat$ is $\varpi^\flat$-torsion free.)

If A is completely integrally closed in $A[\frac{1}{\varpi}]$,

then $A^\flat$ is completely integrally closed in $A^\flat[\frac{1}{\varpi^\flat}]$.

uses only multiplicative structure of rings

§2. $R_0 \to R_1 \to R_2 \to \cdots$: perfectoid tower arising from $(R, I_0 = (f_0))$

$\to \widehat{R_\infty}$: perfectoid with "ϖ^p" $= f_0$

Def 2.1 (Small tilts ; recall)

$$R_0^{s.b} \underset{\mathrm{def}}{=} \varprojlim \left(R_0/I_0R_0 \xleftarrow{F_0} R_1/I_0R_1 \xleftarrow{F_1} \cdots \right)$$

Prop-Def 2.2 (H.-Ishiro-Shimomoto)

①
$$\begin{tikzcd} R_i/I_0R_i \arrow[r, hook] \arrow[dr, "\cong"'] & \widehat{R_\infty}/I_0\widehat{R_\infty} \\ & R_i+I_0\widehat{R_\infty}/I_0\widehat{R_\infty} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

② $\varprojlim \left(R_0+I_0\widehat{R_\infty} \xleftarrow{x\mapsto x^p} R_1+I_0\widehat{R_\infty} \xleftarrow{x\mapsto x^p} \cdots \right) \to R_0^{s.b}$

is an isom. of monoids.

$\#^{(0)}$:
$$\begin{tikzcd} R_0^{s.b} \arrow[d, phantom, "\cup"] & \varprojlim_{i\geq 0}(R_i+I_0\widehat{R_\infty}) \arrow[l, "\cong"'] \arrow[r, "pr_0"] & R_0+I_0\widehat{R_\infty} \arrow[d, phantom, "\cup"] \\ \exists f_0^{s.b} \arrow[rr, mapsto] & & f_0 \end{tikzcd}$$

Rem
$$\begin{tikzcd} (\widehat{R_\infty})^b \arrow[r, "\#"] & R_\infty^b \\ R_i^{s.b} \arrow[u, hook] \arrow[r, "\#^{(i)}"'] & R_i+I_0\widehat{R_\infty} \arrow[u, hook] \end{tikzcd}$$

Thm 2.3 (Main Thm A) (H.-Ishiro-Shimomoto)

Assume R is f_0-torsion free

($\Longleftrightarrow$ R_i is f_0-torsion free for $\forall i \geq 0$.

$\Longleftrightarrow$ $R_i^{s.b}$ is $f_0^{s.b}$-torsion free for $\forall i \geq 0$.)

If R_i is completely integrally closed in $R_i[\frac{1}{f_0}]$ for $\forall i \geq 0$,

then $R_i^{s.b}$ is (completely) integrally closed in $R_i^{s.b}[\frac{1}{f_0^{s.b}}]$ for $\forall i \geq 0$.

if $R_i^{s.b}$ is Noetherian.

Main Thm B

$$\begin{array}{ccccccccc}
R = W(k)[[x_2, \cdots, x_d]] & \hookrightarrow & R_1 & \hookrightarrow & \cdots & \hookrightarrow & R_n & \hookrightarrow & \cdots \\
\text{fin.} \downarrow & & & & & & \downarrow & & \\
S & \hookrightarrow & \cdots & \hookrightarrow & \boxed{S_N \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow \cdots} & - & \circledast & &
\end{array}$$

$\exists \varepsilon \in (0,1) \cap \mathbb{Q}$ $\exists N \geq 0$

S: normal local domain s.t. $R[\frac{1}{p}] \hookrightarrow S[\frac{1}{p}]$: étale

S_n: the int. closure of R_n in $R_n \otimes_R S[\frac{1}{p}]$

$\circledast$ is a perfectoid tower arising from $(S_N, (p^{\varepsilon}))$.

Rem Its proof relies on results in ramification theory by Faltings & Andreatta

Cor $S_i^{s.b}$ is normal for $\forall i \geq N$.