Fontaineのモノイダル写像の改良について

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§1. Fontaine's map

§2. Refined Fontaine's map.

$\boxed{\text{Fix a prime number } p}$

$$
\begin{tikzcd}
\mathbb{Z}_p \arrow[d, phantom, "\cap"] & \mathbb{F}_p[[T]] \arrow[l, dashed, "\sharp^{(0)}"] \arrow[l, dashed, leftrightarrow, bend right=40, "?"'] \arrow[d, phantom, "\cap"] \\
\mathbb{Z}_p[p^{1/p}] \arrow[d, phantom, "\cap"] & \mathbb{F}_p[[T]][T^{1/p}] \arrow[l, dashed, "\sharp^{(1)}"] \arrow[d, phantom, "\cap"] \\
\mathbb{Z}_p[p^{1/p^2}] \arrow[d, phantom, "\cap"] & \mathbb{F}_p[[T]][T^{1/p^2}] \arrow[l, dashed, "\sharp^{(2)}"] \arrow[d, phantom, "\cap"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\widehat{\mathbb{Z}_p[p^{1/p^\infty}]} & \widehat{\mathbb{F}_p[[T]][T^{1/p^\infty}]} \arrow[l, "\sharp"]
\end{tikzcd}
$$

References

[1] F. Andreatta, Generalized ring of norms and generalized (φ, Γ)-modules, Ann. Sci. École Norm. Sup. (4) 39 (2006), 599–647.

[2] K. Eto, J. Horiuchi, and K. Shimomoto, Some ring-theoretic properties of rings via Frobenius and monoidal maps, Tokyo J. Mathematics 47 (2024), 395–409.

§1 A : ring

Def (Tilts)

$$A^\flat \underset{\text{def}}{=} \varprojlim \left(A/pA \xleftarrow{\text{Frob}} A/pA \xleftarrow{\text{Frob}} \cdots \right)$$

This is a perfect $\mathbb{F}_p$-algebra,

i.e., Frob. on $A^\flat$ is bij.

Prop-Def (Fontaine's monoidal map)

If A is p-adically complete,

$$\begin{tikzcd} \varprojlim_{x \mapsto x^p} A \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}_0"'] & A^\flat \arrow[dl, "\#"] \arrow[dl, phantom, bend right, "\circlearrowright"] \\ A & \end{tikzcd}$$

as (topological) monoids.

$$a = (\bar{a}_n)_{n \geq 0} \in A^\flat \longmapsto a^\# = \lim_{k \to \infty} a_k^{p^k} \in A$$

Ex $A = W(k)$, k : perfect field of char. p

$$\begin{tikzcd} A = W(k) & \\ A^\flat \cong k \arrow[u, "\#"] \arrow[u, bend right, "{[\cdot]}"'] \arrow[u, phantom, shift right=3, "\circlearrowright"] \end{tikzcd}$$

$[\cdot]$: Teichmüller map

$\sharp : A^\flat \to A$ plays a fundamental role in perfectoid geometry (Kedlaya-Liu, Scholze)

Thm 1 (Eto-Horiuchi-Shimomoto)

Assume A is p-torsionfree perfectoid

with $\pi \in NZD(A)$

$p \in \pi^p A$

$\pi^\flat := (\pi, \pi^{1/p}, \pi^{1/p^2}, \cdots) \in \varprojlim_{x \mapsto x^p} A \xrightarrow{\cong} A^\flat$

If A is (completely) integrally closed in $A[\frac{1}{\pi}]$

then $A^\flat$ is (completely) integrally closed in $A^\flat[\frac{1}{\pi^\flat}]$

Rem A is completely integrally closed in $A[\frac{1}{\pi}]$

$\Leftrightarrow$ $^\forall x \in A[\frac{1}{\pi}]$

$^\exists c > 0$ s.t. $\pi^c x^n \in A$ for $^\forall n \geq 0$

$\Rightarrow x \in A$ using only the multiplicative structure of the ring.

§2 $(R, \mathfrak{m}, k)$: an unramified complete RLR

with $k \supset \mathbb{F}_p$ perfect

$\cong W(k)[[x_2, \cdots, x_d]]$

$$R_0 = R \hookrightarrow R_1 := R[p^{1/p}, x_2^{1/p}, \cdots, x_d^{1/p}] \hookrightarrow \cdots \hookrightarrow R_n := R[p^{1/p^n}, x_2^{1/p^n}, \cdots, x_d^{1/p^n}] \hookrightarrow \cdots$$

Rem $\{R_n\}_{n \geq 0}$ is a perfectoid tower.

(Ishiro-Nakazato-Shimomoto)

→ ① $\widehat{R_\infty} := \widehat{\bigcup_{n \geq 0} R_n}^{p}$ is perfectoid.

② (Small tilts)

$$R_n^{s.b} := \varprojlim \left(R_n/pR_n \xleftarrow{F_n} R_{n+1}/pR_{n+1} \xleftarrow{F_{n+1}} \cdots \right)$$

$$x^p \longmapsfrom x$$

$$\cong k[[(p^\flat)^{1/p^n}, (x_2^\flat)^{1/p^n}, \cdots, (x_d^\flat)^{1/p^n}]]$$

Key property : $\boxed{p\widehat{R_\infty} \cap R_n = pR_n}$

$$\to \quad R_n/pR_n \xrightarrow{\cong} R_n + p\widehat{R_\infty} / p\widehat{R_\infty} \twoheadleftarrow R_n + p\widehat{R_\infty}$$

Prop-Def (Refined Fontaine's map)

$$
\begin{tikzcd}
R_\infty^\flat \arrow[rr, bend left, "\#"] & \varprojlim\limits_{x\mapsto x^p} \widehat{R_\infty} \arrow[l, "\cong"'] \arrow[r, "pr_0"] & \widehat{R_\infty} \\
\cup & & \cup \\
\vdots & & \vdots \\
\cup & & \cup \\
R_n^{s.b} \arrow[rr, bend right, "\#^{(n)}"'] & \varprojlim \left(R_n + p\widehat{R_\infty} \leftarrow R_{n+1} + p\widehat{R_\infty} \leftarrow \cdots \right) \arrow[l, "\cong"'] \arrow[r, "pr_0"] & R_n + p\widehat{R_\infty}
\end{tikzcd}
$$

$$R_n + p\widehat{R_\infty} \ni x^p \mapsfrom x \in R_{n+1} + p\widehat{R_\infty}$$

Rem This constuction of $\#^{(n)}$ applies to "perfectoid-like" towers.

We introduce an example with applications.

R : as above

$R \hookrightarrow S$: module-finite ext.

of normal local domains

with $R[\frac{1}{p}] \hookrightarrow S[\frac{1}{p}]$ finite étale

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d] & S \arrow[d] & \\
R_n \arrow[r] \arrow[dr] & R_n \otimes_R S \arrow[r] \arrow[d, dashed] & (R_n \otimes_R S)[\frac{1}{p}] \\
& S_n \arrow[ur] &
\end{tikzcd}$$

${\color{red}S_n}$: the integral closure of R_n in $(R_n \otimes_R S)[\frac{1}{p}]$

<u>Thm 2</u> (Faltings, Andreatta)

$^{\exists}{\color{red}\varepsilon} = \varepsilon(S) \in (0,1) \cap \mathbb{Q}$

$^{\exists}{\color{red}N} = N(S, \varepsilon) \geq 0$ s.t.

the const of $\#^{(n)}$ applies to the tower $\{S_n\}_{n \geq {\color{red}N}}$ by replacing $p\widehat{R_\infty}$ with $p^{\color{red}\varepsilon}\widehat{S_\infty}$

<u>Rem</u> Its proof relies on ramification theory.

$\to \ \sharp^{(n)} : \mathcal{S}_n^{s.b} \to \mathcal{S}_n + p^{\varepsilon}\widehat{\mathcal{S}_\infty}$.

This carries over the normality of $\mathcal{S}_n$ to $\mathcal{S}_n^{s.b}$:

Thm 3 (H.-Ishiro-Shimomoto)

$\forall n \geq N$, $\mathcal{S}_n^{s.b}$ is normal.

Its proof is reduced to show that

$\mathcal{S}_n^{s.b}$ is completely integrally closed in $\mathcal{S}_n^{s.b}[1/p^b]$,

and we proceed as in the proof of Thm 1.

Problem ① Find other interesting examples.

② Find other applications.

e.g., refine the following results :

- $\mathrm{Spa}(A[\frac{1}{\pi}], A) \xrightarrow{\approx} \mathrm{Spa}(A^\flat[\frac{1}{\pi^\flat}], A^\flat)$

(Kedlaya-Liu, Scholze)

- $$\begin{array}{ccc} \mathrm{Spec}(A) & & \mathrm{Spec}(A^\flat) \\ \cup & & \cup \\ \mathrm{Spec}_{perfd}(A) & \approx & \mathrm{Spec}_{perfd}(A^\flat) \end{array}$$

(Dine-Ishizuka)