An introduction to "Regular rings and perfect(oid) algberas"

(「Regular rings and perfect(oid) algebras」の紹介)

Graduate School of Mathematics, Nagoya University
Kazuki HAYASHI *

Original version: August 28, 2023
Updated: June 25, 2026

Abstract

This is a note for the 18-th Summer School on Commutative Algebra at the Tokyo Institute of Technology to introduce the paper [BIM19].

Contents

Introduction

The paper [BIM19] explores some homological properties of perfect(oid) algebras over commutative noetherian rings. One of its main results is Kunz's theorem in mixed characteristic.

Recall that Kunz's theorem (see Theorem 1.1) asserts that a noetherian $\mathbb{F}_p$-algebra R is regular if and only if the Frobenius map $R \to R$ is flat. This result can be reformulated as the following assertion: such an R is regular precisely when there exists a faithfully flat map $R \to A$ with A perfect (see Proposition 1.6). The p-adic generalization is as follows:

Theorem A (Theorem 1.10)**.** *Let R be a noetherian ring such that p lies in the Jacobson radical of R (for instance, R could be p-adically complete). Then R is regular if and only if there exists a faithfully flat map $R \to A$ with A perfectoid.*

*E-mail: hazuki0694@gmail.com

§ 1 is devoted to Kunz's theorem in both positive characteristic and mixed characteristic.

In § 2, we introduce results on the finiteness of flat dimension. In fact, the original proof of Theorem A in [BIM19] is based on the results in this section.

In § 3, we focus on two algebras: the absolute integral closure R^+ and the perfect closure R_{perf}. We also introduce the notions of *proregular sequences* and *weakly proregular sequences*. It will turn out that, under suitable hypotheses, systems of parameters of R forrm weakly proregular sequences on R^+ and on R_{perf} (see Proposition 3.9). As a consequence, we obtain a criterion of regularity in terms of the vanishing of $\mathrm{Tor}_i^R(R^+, R/\mathfrak{m}_R)$ or $\mathrm{Tor}_i^R(R_{\mathrm{perf}}, R/\mathfrak{m}_R)$ (see Theorem 3.11).

Notations and conventions.

- All rings are assumed to be commutative and contains unity.
- Let p denote a fixed prime number.

1 Kunz's theorem

In this section we give a quick but reasonably detailed overview of the proof of Kunz's theorem in the mixed characteristic case.

First, in §1.1, we recall classical Kunzs theorem and some of its applications. This subsection ends with reformulation of Kunz's theorem

After discussing the proof of this reformulation in positive characteristic, we proceed in § 1.2 to its generalization to mixed characteristic, using the notion of perfectoid rings.

1.1 Positive characteristic case

Throughout this subsection, let R denote a noetherian $\mathbb{F}_p$-algebra. Let us first recall Kunz's theorem.

Theorem 1.1 ([Kun69, Theorem 2.1 and Corollary 2.7])**.** *The following conditions are equivalent.*

(1) *R is regular.*

(2) *The absolute Frobenius $\varphi\colon R \to R$ is flat.*

Sketch. We may assume that R is complete local (here we use the fact that a local homomorphism $A \to B$ of noetherian local rings is flat if and only if so is $\widehat{A} \to \widehat{B}$, which is a consequence of local criterion of flatness). Let $k := R/\mathfrak{m}_R$.

(1) $\Rightarrow$ (2): By Cohen's structure theorem, we may assume that $R = k[\![X_1, \ldots, X_n]\!]$, where $n = \dim R$. Then the canonical injection $R^p \hookrightarrow R$ can be decomposed as

$$R^p = k^p[\![X_1^p, \ldots, X_n^p]\!] \hookrightarrow k[\![X_1^p, \ldots, X_n^p]\!] \hookrightarrow k[\![X_1, \ldots, X_n]\!] = R.$$

Since $k[\![X_1, \ldots, X_n]\!]$ is free over $k[\![X_1^p, \ldots, X_n^p]\!]$ on the basis $\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid 0 \leq \alpha_i \leq p-1\}$ and since the flatness of $k^p \hookrightarrow k$ implies that of $k^p[\![X_1^p, \ldots, X_n^p]\!] \hookrightarrow k[\![X_1^p, \ldots, X_n^p]\!]$, it follows that $R^p \hookrightarrow R$ is flat.

(2) $\Rightarrow$ (1): Let $x_1, \ldots, x_r$ be a minimal basis of $\mathfrak{m}_R$. Then by Cohen's structure theorem, we have a surjection

$$S := k[\![X_1, \ldots, X_r]\!] \twoheadrightarrow R, \quad X_i \mapsto x_i.$$

Let $\mathfrak{a}$ be the kernel. Then, for each p-power $q = p^\nu$, the surjection induces the short exact sequence

$$0 \longrightarrow (\mathfrak{a} + \mathfrak{m}_S^{[q]})/\mathfrak{m}_S^{[q]} \longrightarrow S/\mathfrak{m}_S^{[q]} \longrightarrow R/\mathfrak{m}_R^{[q]} \longrightarrow 0.$$

Using the notion of independence in the sense of Lech, we can prove that $l_S(S/\mathfrak{m}_S^{[q]}) = q^r = l_R(R/\mathfrak{m}_R^{[q]}) = l_S(R/\mathfrak{m}_R^{[q]})$, so that $(\mathfrak{a} + \mathfrak{m}_S^{[q]})/\mathfrak{m}_S^{[q]} = 0$, i.e., $\mathfrak{a} \subset \mathfrak{m}_S^{[q]}$. Since this holds true for any p-power $q = p^\nu$, one has

$$\mathfrak{a} \subset \bigcap_{\nu > 0} \mathfrak{m}_S^{[q]} = (0).$$

Thus $R \cong S = k[\![X_1, \dots, X_r]\!]$, which is a regular local ring. $\square$

Let us mention some applications of Kunz's theorem. The fact that a localization of a regular local ring is again regular is proved using the Auslander–Buchbaum–Serre theorem for regular local rings. In positive characteristic, however, this fact is an immediate consequence of Kunz's theorem:

Corollary 1.2 ([Kun69, Corollary 2.2])**.** *If R is a regular local ring, then so is $R_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*

In addition, Kunz's theorem yields the following result about excellence of $\mathbb{F}_p$-algebras.

Theorem 1.3 ([Kun76, Theorem 2.5])**.** *If the absolute Frobenius $\varphi\colon R \to R$ is finite, then R is excellent.*

The following corollay leads to our reformulation of Kunz's theorem.

Corollary 1.4. *If R is a regular, then $R \to R_{\mathrm{perf}} := \varinjlim_\varphi R$ is faithfully flat.*

Proof. It suffices to show that each $\varphi^n\colon R \to R$ is faithfully flat (cf. [Sta, Tag 090N]). Since R is a regular $\mathbb{F}_p$-algebra, φ (hence φ^n) is flat by Kunz's theorem. Moreover, given a maximal ideal $\mathfrak{m} \subset R$, we have $\varphi^n(m)R \subset \mathfrak{m}^{p^n} R \subset \mathfrak{m} \neq R$. Thus we conclude that $\varphi^n\colon R \to R$ is faithfully flat. $\square$

We will show that the converse holds, and at the same time reformulate Kunz's theorem. Let us start with the following lemma.

Lemma 1.5 (cf. [BIM19, Lemma 3.2])**.** *Let S be a perfect $\mathbb{F}_p$-algebra, and $\boldsymbol{x} = x_1, \dots, x_n$ a sequence of elements in S.*
(1) $\sqrt{\boldsymbol{x}S} = (x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty}) := \bigcup_{e>0} (x_1^{1/p^e}, \dots, x_n^{1/p^e})$.
(2) $\operatorname{fd}_S(S/\sqrt{\boldsymbol{x}S}) \leq n$.

Proof. (1) Straightforward.

(2) We proceed by induction on n.

$\underline{n=1}$ Relabel $x = x_1$ for visual convenience. It suffices to check that the ideal $I := (x^{1/p^\infty}) \subset S$ is flat as an S-module (then $0 \to I \to S \to S/I \to 0$ is a flat resolution of S/I). Observe that the morphism of direct systems[1]

$$\begin{tikzcd}
S \arrow[r, "x^{1-\frac{1}{p}}"] \arrow[d, "x"] & S \arrow[r, "x^{\frac{1}{p}-\frac{1}{p^2}}"] \arrow[d, "x^{1/p}"] & S \arrow[r, "x^{\frac{1}{p^2}-\frac{1}{p^3}}"] \arrow[d, "x^{1/p^2}"] & \cdots \\
(x) \arrow[r, phantom, "\subset"] & (x^{1/p}) \arrow[r, phantom, "\subset"] & (x^{1/p^2}) \arrow[r, phantom, "\subset"] & \cdots
\end{tikzcd}$$

induces the morphism of direct limits

$$\psi\colon \varinjlim \left(S \xrightarrow{x^{1-\frac{1}{p}}} S \xrightarrow{x^{\frac{1}{p}-\frac{1}{p^2}}} S \to \cdots \right) \to (x^{1/p^\infty}) = I.$$

[1]The notation $x^{\frac{1}{p^e}-\frac{1}{p^{e+1}}}$ makes sense because $\frac{1}{p^e} - \frac{1}{p^{e+1}} = \frac{p-1}{p^{e+1}}$.

The surjectivity is clear, and we can check the injectivity using the fact that S is reduced.

$\underline{n > 1}$ Set $S' := S/\sqrt{x_1 S}$, and let $\boldsymbol{x}'$ be the image of the sequence $x_2, \ldots, x_n$ in S'. Then S' is also perfect, and thus by the induction hypothesis, we obtain

$$\mathrm{fd}_S(S/\sqrt{\boldsymbol{x}S}) \leq \mathrm{fd}_{S'}(S'/\sqrt{\boldsymbol{x}'S'}) + 1 \leq (n-1) + 1 = n,$$

(see [AF91, 4.2 Corollary (b) (F)] or [Sta, Tag 066K], for the first inequality). □

Now one can reformulate Kunz's theorem as the following assertion:

Proposition 1.6. *The following conditions are equivalent.*

(1) *R is regular.*
(2) *$R \to R_{\mathrm{perf}}$ is faithfully flat.*
(3) *There exists a faithfully flat ring homomorphism $R \to A$ with A perfect.*

Proof. (1) $\Longrightarrow$ (2): Corollary 1.4.

(2) $\Longrightarrow$ (3): Trivial.

(3) $\Longrightarrow$ (1): Pick $\mathfrak{p} \in \operatorname{Spec} R$. Since $R \to A$ is faithfully flat, there exists $P \in \operatorname{Spec} A$ such that $P \cap R = \mathfrak{p}$. Then the induced local homomorphism $R_\mathfrak{p} \to A_P$ is faithfully flat. Thus we may assume that R, A are local and that the flat ring homomorphism $R \to A$ is local. Set $k := R/\mathfrak{m}_R$. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a system of parameters of R ($n = \dim R$). Then Lemma 1.5 yields

$$\mathrm{fd}_A(A/\sqrt{\boldsymbol{x}A}) \leq n. \tag{1.1}$$

Hence, if $i > n$,

$$\begin{aligned} 0 &= \mathrm{Tor}_i^A(k \otimes_R A, A/\sqrt{\mathfrak{m}_R A}) \\ &\overset{\text{flat}}{=} \mathrm{Tor}_i^R(k, A/\sqrt{\mathfrak{m}_R A}) \\ &= \mathrm{Tor}_i^R(k,k)^{\oplus I} \end{aligned}$$

where $A/\sqrt{\boldsymbol{x}A} \cong k^{\oplus I}$ (since $A/\sqrt{\boldsymbol{x}A} = A/\sqrt{\sqrt{\boldsymbol{x}R}A} = A/\sqrt{\mathfrak{m}A}$ is a k-vector space). Since $R \to A$ is a local homomorphism, it follows that $I \neq \emptyset$, so that $\mathrm{Tor}_i^R(k,k) = 0$ for $i > n$. This means that $\operatorname{gl.dim} R = \mathrm{pd}_R\, k \leq n < \infty$. □

Note that the essential part of the proof is the finiteness of flat dimension (1.1).

1.2 Mixed characteristic case

We begin with the definition and some properties of perfectoid rings.

Definition 1.7 ([BIM19, Definition 3.5])**.** We say that a ring A is *perfectoid* if it satisfies the following conditions.

(1) A is p-adically complete.
(2) The absolute Frobenius of the $\mathbb{F}_p$-algebra A/pA is surjective.
(3) The kernel of Fontaine's map $\theta\colon W(A^\flat) \to A$ is principal.
(4) There exist $\pi \in A$ and $u \in A^\times$ such that $\pi^p = pu$.

For the equivalence of this definition and others, see [BMS1, Proposition 3.10]. If A is perfectoid, then the following hold:

- Fontaine's map $\theta\colon W(A^\flat) \to A$ is surjective (see [BMS1, Lemma 3.9]).
- An element $\xi = (\xi_0, \xi_1, \ldots) \in \operatorname{Ker}\theta$ generates $\operatorname{Ker}\theta$ if and only if ξ is distinguished, i.e., $\xi_1 \in (A^\flat)^\times$ (see [BMS1, Remark 3.11]).

Note also that an arbitrary product of perfectoid rings is perfectoid ([BIM19, Example 3.8 (8)]). Indeed, the condition (3) in Definition 1.7 is satisfied because the functor $A \mapsto W(A^\flat)$ commutes with products. The other conditions are obvious.

We now turn to generalizing Kunz's theorem to mixed characteristic. The most important features of perfectoid rings are the following.

Lemma 1.8 ([BIM19, Lemma 3.7]). *Let A be a perfectoid ring.*

(1) *The $\mathbb{F}_p$-algebra $\overline{A} := A/\sqrt{pA}$ is perfect.*
(2) *The ideal $\sqrt{pA} \subset A$ is a flat A-module.*

Proof. (1) We first show that the element π appearing in Definition 1.7 can be assumed to admit a compatible system of p-power roots $\{\pi^{1/p^n}\}_{n\geq 1}$. Let $\xi = (\xi_0, \xi_1, \dots) \in W(A^\flat)$ be a generator of $\operatorname{Ker}\theta$, and set $\pi \in A$ to be the image of $[\xi_0]$. Then π satisfies the condition (4) in Definition 1.7 (here we use that ξ is distinguished) and admits a compatible system of p-power roots, namely the images of $[\xi_0^{1/p^{n+1}}]$.

Since $(p) = (\pi^p)$, one has $\sqrt{pA} = (p^{1/p^\infty}) = (\pi^{1/p^\infty})$, and so it suffices to show that $A/(\pi^{1/p^\infty})$ is perfect. The isomorphism $W(A^\flat)/(\xi) \xrightarrow{\sim} A$ and the definition of our π yield

$$A/(\pi^{1/p^\infty}) \cong W(A^\flat)/(\xi, [\xi_0^{1/p^\infty}]) \cong \frac{W(A^\flat)/(p)}{(\xi, [\xi_0^{1/p^\infty}])/(p)} \cong A^\flat/\left(\xi_0^{1/p^\infty}\right) \overset{\text{Lemma 1.5 (1)}}{=} A^\flat/\sqrt{(\xi_0)}.$$

This ring is perfect since $A^\flat$ is perfect.

(2) We prove only in the case where A is p-torsion free. (The general case is difficult.) We check that $\operatorname{fd}_A(\overline{A}) \leq 1$; this is equivalent to showing that $\sqrt{pA}$ is flat. Since A is p-torsion free and $\pi^p = pu$ for some $u \in A^\times$, we can see that each π^{1/p^e} is a non-zero-divisor of A. Thus each $\varpi^{1/p^e}A$ is isomorphic to A, hence is free. The directed union (π^{1/p^∞}) is also flat. $\square$

By this lemma, we deduce the desired finiteness of flat dimension. For the sake of the later applications, we give this result in more general setting. We say that a ring A is (*positive characteristic p and*) *perfect modulo a flat ideal* if there exists an ideal $I \subset A$ containing p such that A/I is perfect. Lemma 1.8 shows that any perfectoid ring is perfect modulo a flat ideal.

Lemma 1.9. *Let A be a ring that is perfect modulo a flat ideal I, and set $\overline{A} := A/I$. If $J \subset A$ is an ideal containing I such that $J\overline{A} = \sqrt{\boldsymbol{x}\overline{A}}$ for some sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\overline{A}$. Then* $\operatorname{fd}_A(A/J) \leq n+1$.

Proof. $\operatorname{fd}_A(A/J) \leq \operatorname{fd}_{\overline{A}}(\overline{A}/J\overline{A}) + \operatorname{fd}_A \overline{A} \overset{\text{Lemma 1.8}}{\leq} n+1.$ $\square$

Now we can prove the following mixed characteristic generalization of Kunz's theorem.

Theorem 1.10 ([BIM19, Theorem 4.7]). *Let R be a noetherian ring with $p \in \operatorname{rad} R$. Then the following conditions are equivalent.*

(1) *R is regular.*
(2) *There exists a faithfully flat ring homomorphism $R \to A$ with A perfectoid.*

Proof. (2) $\Rightarrow$ (1): Because of Lemma 1.8 and Lemma 1.9, the argument similar to that in Proposition 1.6 works.

(1) $\Rightarrow$ (2): Assume that R is regular with $p \in \operatorname{rad} R$. We must construct a faithfully flat ring homomorphism $R \to A$ with perfectoid.

<u>Step 1. Reduction to the case where R is complete local.</u>

Assume that, for each $\mathfrak{m} \in \operatorname{Max} R$, we obtain a faithfully flat ring homomorphism $\widehat{R}_{\mathfrak{m}} \to A(\mathfrak{m})$ with $A(\mathfrak{m})$ perfectoid. Consider the resulting ring homomorphism

$$R \to \prod_{\mathfrak{m} \in \operatorname{Max} R} \widehat{R}_{\mathfrak{m}} \to \prod_{\mathfrak{m} \in \operatorname{Max} R} A(\mathfrak{m}).$$

As R is noetherian, an arbitrary product of flat R-modules is flat ([岩永佐藤, 命題 8-2-7]), so the above map is flat. Moreover, it is also faithfully flat: the induced map $\operatorname{Spec}\left(\prod_{\mathfrak{m} \in \operatorname{Max} R} A(\mathfrak{m})\right) \to \operatorname{Spec} R$ is open (by flatness), and thus its image is generization-closed. Moreover, the image contains all closed points by construction. As a product of perfectoid rings is perfectoid, we have constructed the desired covers.

STEP 2. Since we have seen the positive characteristic case in Proposition 1.6, it remains the case of mixed characteristic $(0, p)$ (we note that $p \in \operatorname{rad} R$). By [BouAC89, IX, App., Theorem 1, Corolally], there exists a *gonflement* $R \to S$ such that the residue field of S is an algebraicly closure of $R/\mathfrak{m}_R$. Then $R \to S$ is faithfully flat ([BouAC89, IX, App., Proposition 2, b)]) and S is also regular ([BouAC89, IX, App., Proposition 2, Corollary)]). Thus we may replace R by $\widehat{S}$, hence may assume that $R/\mathfrak{m}_R$ is perfect. Then, by Cohen's structure theorem,

$$R = \begin{cases} W(k)[\![X_2, \dots, X_d]\!] & \text{if } R \text{ is unramified,} \\ W(k)[\![X_1, \dots, X_d]\!]/(p - f) & \text{if } R \text{ is ramified,} \end{cases}$$

where $f \in (x_1, \dots, x_d)^2 \setminus (p)$. We take

$$A := \begin{cases} \left(W(k)[p^{1/p^\infty}][\![X_2^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]\right)_p^\wedge & \text{if } R \text{ is unramified,} \\ \left(W(k)[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]/(p - f)\right)_p^\wedge & \text{if } R \text{ is ramified.} \end{cases}$$

Indeed, A is perfectoid and $R \to A$ is faithfully flat:

(perfectoid): The unramified case follows as in the case $k = \mathbb{F}_p$ (hence $W(k) = \mathbb{Z}_p$). The ramified case is due to Shimomoto [Shi16, Proposition 4.9].

(faithfully flat): Observe that A is the p-adic completion of

$$S := \begin{cases} W(k)[p^{1/p^\infty}][\![X_2^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!] = \bigcup_{n>0} W(k)[p^{1/p^n}][\![X_2^{1/p^n}, \dots, X_d^{1/p^n}]\!], \\ W(k)[\![X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]\!]/(p - f) = \bigcup_{n>0} W(k)[\![X_1^{1/p^n}, \dots, X_d^{1/p^n}]\!]/(p - f). \end{cases}$$

In the ramified case, $R \to A$ is faithfully flat by [Bha18, Proposition 5.1] and the fact that $R/pR \to S/pS \xrightarrow{\sim} A/pA$ is faithfully flat. $\square$

2 Criteria for finite flat dimension

2.1 Local cohomology

Local cohomology is an important tool in homological algebra because of the Grothendieck (non-)vanishing results related to the depth and dimension of a finitely generated module over a Noetherian local ring (cf. [BH, Theorem 3.5.7]). Let us briefly the definition of local cohomology.

Let R be a ring, and $\mathfrak{a} \subset R$ an ideal. For an R-module M, the $\mathfrak{a}$-torsion submodule of M is defined by

$$\Gamma_{\mathfrak{a}}(M) \stackrel{\text{def}}{=} \bigcup_{q \geq 0} 0 :_M \mathfrak{a}^q \cong \varinjlim_q \operatorname{Hom}_R(R/\mathfrak{a}^q, M).$$

Then $\Gamma_{\mathfrak{a}}(-)$ defines a left exact functor from the category $\mathsf{Mod}\, R$ of R-modules into itself. The right derived functors of $\Gamma_{\mathfrak{a}}(-)$ is called the *local cohomology functors*, denoted by $H^i_{\mathfrak{a}}(-)$.

Remark 2.1. On the other hand, the completion functor $\Lambda^{\mathfrak{a}}(-)$ in commutative algebra, which is given by

$$\Lambda^{\mathfrak{a}}(M) \overset{\text{def}}{=} \varprojlim_q (R/\mathfrak{a}^q \otimes_R M),$$

is formally dual to the functor $\Gamma_{\mathfrak{a}}(-)$. However, it is neither left nor right exact in general. In order to study the left derived functors $L_i\Lambda^{\mathfrak{a}}$ J. P. C. Greenless and J. P. May [GM92] introduced the notion of a "proregular sequence," which we will introduce in 3.1.

For the sake of our applications in this section, we just remark here the following result:

Theorem 2.2. *Let R be a noetherian ring, and M an R-module. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements in R, and set $\mathfrak{a} := (\boldsymbol{x})$. Then*

$$H^i_{\mathfrak{a}}(M) \cong \check{H}^i(\boldsymbol{x}; M) := H^i(\check{C}(\boldsymbol{x}) \otimes_R M).$$

For the proof, see [BH, Theorem 3.5.6]. Note that this implies the following result.

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M an R-module. Then*

$$s(M) := \sup\{t \mid H^t_{\mathfrak{m}}(M^{\vee})\} \neq 0\} \leq \dim R.$$

where $M^{\vee} := \mathrm{Hom}_R(M, E_R(k))$ is the Matlis dual.

Proof. If $\boldsymbol{x}$ is a system of parameters of R, then $\check{C}^i(\boldsymbol{x}) = 0$ for $i > \dim R$, and thus $H^i_{\mathfrak{m}}(M) \overset{\text{Theorem 2.2}}{\cong} H^i(\check{C}(\boldsymbol{x}) \otimes_R M) = 0$ for $i > \dim R$, which confirms the assertion. □

In addition, we need the following fact, which concerns a rigidity.

Fact 2.4 ([CIM19, Proposition 3.3])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M an R-module. Set $s(M)$ be as in Theorem 2.2. If $\mathrm{Tor}^R_i(M, k) = 0$ for some $i \geq s(M)$, then $\mathrm{Tor}^R_j(M, k) = 0$ for all $j \geq i$.*

2.2 Results in [BIM19]

Theorem 2.5 ([BIM19, Theorem 2.1])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and S an R-algebra containing an ideal J with $\mathfrak{m}S \subseteq J$ and $\mathrm{fd}_S(S/J) < \infty$. Let $d \geq \mathrm{fd}_S(S/J)$ be an integer, and U an S-module with $JU \neq U$. Let M be an R-module, and $s \geq 0$ an integer. If $\mathrm{Tor}^R_j(U, M) = 0$ for $j \geq s$, then $\mathrm{Tor}^R_j(k, M) = 0$ for $j \geq s + d$.*

Theorem 2.6 ([BIM19, Theorem 4.1, Remark 4.3])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring with $\mathrm{char}(k) = p$, and A an R-algebra with $\mathfrak{m}A \neq A$. Assume that $R \to A$ factors through an R-algebra S that is perfect modulo a flat ideal. Let M be an R-module. If $\mathrm{Tor}^R_j(A, M) = 0$ for $j \gg 0$, then $\mathrm{Tor}^R_j(k, M) = 0$ for $j \gg 0$.*

Proof. Apply Theorem 2.5 when the case:
- $J = \sqrt{\mathfrak{m}S} = \sqrt{\boldsymbol{x}S}$, where $\boldsymbol{x} = x_1, \ldots, x_n$ is a s.o.p. of R.
- $d := n + 1$.
- $U := A$.

Note that if follows from Lemma 1.9 that $\mathrm{fd}_S(S/J) \leq n + 1 = d$. □

3 Applications

3.1 (Weakly) Proregular sequences

We refer to [Sch03] and [SS].

In the early days of local cohomology, A. Grothendieck proved that the local cohomology of a module over a noetherian ring may be computed by a Čech complex (see Theorem 2.2). In [AJL97, Lemma (3.1.1)], J. Lipman et al. generalized this result, stated in Theorem 2.2, to the case where $\boldsymbol{x}$ is a proregular sequence in an arbitrary commutative ring. However, this turned out to be incorrect, as pointed out by P. Schenzel, and Lipman subsequently introduced the notion of a "weakly proregular sequence." Schenzel proved that Čech cohomology coincides with local cohomology if and only if the ideal is generated by a weakly proregular sequence (see [Sch03, Theorem 3.2]), which has its origins in [AJL97] and [GM92].

Observation 3.1. Let R be a ring, and $\boldsymbol{x} = x_1, \dots, x_d$ a sequence of elements in R. If $\boldsymbol{x}$ is an R-regular sequence, then for each $n \geq 1$, $x_1^n, \dots, x_d^n$ is again an R-regular sequence (see [Mat2, Theorem 16.1]) and so,

$$\frac{(x_1^n, \dots, x_{i-1}^n) :_R x_i^n}{(x_1^n, \dots, x_{i-1}^n)} = 0.$$

Due to this observation, we see that proregular sequences, defined as the following, is indeed a generalization of regular sequences.

Definition 3.2. Let R be a ring. We say that a sequence $\boldsymbol{x} = x_1, \dots, x_d$ of elements in R is *proregular* if for any $1 \leq i \leq d$ and any $n \geq 1$, there exists $m \geq n$ such that the multiplication map

$$\frac{(x_1^m, \dots, x_{i-1}^m) :_R x_i^m}{(x_1^m, \dots, x_{i-1}^m)} \xrightarrow{x_i^{m-n}} \frac{(x_1^n, \dots, x_{i-1}^n) :_R x_i^n}{(x_1^n, \dots, x_{i-1}^n)}$$

is zero, i.e.,

$$(x_1^m, \dots, x_{i-1}^m) :_R x_i^m \subset (x_1^n, \dots, x_{i-1}^n) :_R x_i^{m-n}.$$

The following notion will be particularly important.

Definition 3.3 ([Har, Definition, p. 23])**.** Let R be a ring. We say that an inverse system $\{X_n\}_{n\geq 0}$ of R-modules is *prozero* (or *essentially zero, essentially null*) if for any $n \geq 0$, there exists $m \geq n$ such that the transition map $X_m \to X_n$ is zero.

If the inverse system $\{X_n\}_{n\geq 0}$ is prozero, then $\varprojlim_n X_n = 0$. The converse is false (e.g., consider the inverse system $\{p^n\mathbb{Z}_p\}_{n\geq 0}$ of $\mathbb{Z}_p$-modules).

Definition 3.4. Let R be a ring. We say that a sequence $\boldsymbol{x} = x_1, \dots, x_d$ of elements in R is *weakly proregular* if for any $q \geq 1$, the inverse system $\{H_q(x_1^n, \dots, x_d^n)\}_{n\geq 1}$ is prozero.

Remark 3.5. (1) The following implications hold:

$$\text{'regular'} \Longrightarrow \text{'proregular'} \Longrightarrow \text{'weakly proregular.'}$$

The first implication is what we have seen. For the proof of the second implication, we refer to [Sch03, Lemma 2.7], or [SS, Lemma A.2.3] for modules, or [Sch21, Theorem 3.4] for homological approach.

(2) If R is noetherian, then any sequence $\boldsymbol{x} = x_1, \dots, x_d$ in R is proregular. To verify this, we only have to remark that for a fixed integer $n \geq 1$, the increasing sequence $\{(x_1^n, \dots, x_{i-1}^n) :_R x_i^{m-n}\}_{m\geq n}$ of ideals of R is stationary.

Example 3.6. For a length one sequence x, we can see that the following conditions are equivalent.

(1) x is poregular.

(2) x is weakly proregular.

(3) The increasing sequence $0 :_R x \subset 0 :_R x^2 \subset 0 :_R x^3 \subset \cdots$ is stationary.

When these conditions are satisfied, we also say that R *is of bounded xR-torsion.*

Example 3.7. Let $R := \prod_{n>0} \mathbb{Z}/2^n\mathbb{Z}$ and $x := (2, 2, \dots) \in R$.

(1) x is neither weakly proregular nor proregular: given $m > 0$, $a := (\overbrace{0, \dots, 0}^{m}, 1, 1, \dots)$ satisfies $ax^{m+1} = 0$ but $ax^m \neq 0$, hence $0 :_R x^m \subsetneq 0 :_R x^{m+1}$. Thus R is not of bounded xR-torsion.

(2) $x, 1$ is not pro-regular (by (1)).

(3) $1, x$ is pro-regular (hence weakly pro-regular): Since $0 :_R 1 = 0 :_R 1^2 = \cdots R$, it follows that R is of bounded $1R$-torsion. For $n > 0$, $1^n R :_R x^n = R = 1^n R :_R x^{n-n}$, and so we may take $m := n$.

By (2) and (3), a proregular sequence is not permutable without any additional assumption.

We care about these notions because of the following observation.

Lemma 3.8 ([BIM19, Lemma 4.10])**.** *Let R be a noetherian ring, and S an R-algebra. If an ideal $\mathfrak{a} \subset R$ is generated, up to radical, by a sequence whose image in S is weakly proregular, then $H^i_{\mathfrak{a}}(I) = 0$ for $i \geq 1$ and any injective S-module I.*

Proof. By hypothesis, there exists a sequence $\boldsymbol{x}$ in R such that $\sqrt{\boldsymbol{x}R} = \sqrt{\mathfrak{a}}$ and $\boldsymbol{x}S$, the image of the sequence $\boldsymbol{x}$ in S, is weakly proregular. Then we have:

$$H^i_{\mathfrak{a}}(I) \overset{\text{Pf. of [BH, 3.5.6]}}{=} H_i(\check{C}(\boldsymbol{x}) \otimes_R I) = H_i((\check{C}(\boldsymbol{x}) \otimes_R S) \otimes_S I) = H_i(\check{C}(\boldsymbol{x}S) \otimes_S I) \overset{\text{[Sch03, Thm. 3.2]}}{=} 0.$$

∎

3.2 Application to R^+ and R_{perf}

Given a domain R, its *absolute integral closure* R^+ is the its integral closure in an algebraic closure of it field of fractions. When R is of positive characteristic, R^+ contains a subalgebra isomorphic to R_{perf}. When R has mixed characteristic $(0, p)$, the ideal $(p^{1/p^\infty})R^+$ is flat, and the quotient ring $R^+/(p^{1/p^\infty})$ is a perfect $\mathbb{F}_p$-algebra. Their most important features for the aim are the following.

Proposition 3.9 ([BIM19, Proposition 4.11])**.** *Let R be an excellent local domain, and $\boldsymbol{x}$ a system of parameters of R.*

(1) *If R has positive characteristic, then $\boldsymbol{x}$ is weakly proregular in R_{perf} and in R^+.*

(2) *If R has mixed characteristic and $\dim R \leq 3$, then $\boldsymbol{x}$ is weakly proregular in R^+.*

Sketch. The assertions for R^+ hold if R^+ is a balanced big Cohen–Macaulay algebra (since a regular sequence is a weakly proregular sequence). This is indeed the case where:

- R has positive characteristic, as proved by M. Hochster and C. Huneke [HH92, Theorem 1.1].
- R has mixed characteristic and $\dim R \leq 2$, as well-known.

It remains to the case where R has mixed characteristic and $\dim R = 3$. But, even in this case, we have a useful result of C. Heitmann [Hei05, Theorem 0.1] (see also [Hei02]), and we can prove the assertion.

We omit the proof for R_{perf}. ∎

Corollary 3.10 ([BIM19, Corollary 4.12]). *Let $(R, \mathfrak{m}, k)$ be an excellent local domain, and $(-)^\vee := \operatorname{Hom}_R(-, E_R(k))$ the Matlis dual.*

(1) *If R has positive characteristic, then $H^i_\mathfrak{m}((R^+)^\vee) = H^i_\mathfrak{m}((R_{\mathrm{perf}})^\vee) = 0$ for $i \geq 1$.*
(2) *If R has mixed characteristic and $\dim R \leq 3$, then $H^i_\mathfrak{m}((R^+)^\vee) = 0$ for $i \geq 1$.*

Proof. By adjunction, the R^+-module $(R^+)^\vee$ and the R_{perf}-module $(R_{\mathrm{perf}})^\vee$ are injective. Thus the desired result follows from Proposition 3.9 and Lemma 3.8. □

Theorem 3.11 ([BIM19, Theorem 4.13]). *Let $(R, \mathfrak{m}, k)$ be an excellent local domain. Then R is regular if any one of the following conditions.*

(1) *R has positive characteristic and $\operatorname{Tor}^R_i(R_{\mathrm{perf}}, k) = 0$ for some $i \geq 1$;*
(2) *R has positive characteristic and $\operatorname{Tor}^R_i(R^+, k) = 0$ for some $i \geq 1$;*
(3) *R has mixed characteristic, $\dim R \leq 3$, and $\operatorname{Tor}^R_i(R^+, k) = 0$ for some $i \geq 1$.*

Remark 3.12. I. M. Aberbach and J. Li [AL08, Corollary 3.5] have proved parts (1) and (2), using different methods.

Proof of Theorem 3.11. In all cases, it follows from Corollary 3.10 and Fact 2.4 that $\operatorname{Tor}^R_j(R^+, k)$, respectively, $\operatorname{Tor}^R_j(R_{\mathrm{perf}}, k)$, is zero for each $j \geq i$. When R has positive characteristic, R^+ contains R_{perf}; in mixed characteristic, R^+ is perfect modulo a flat ideal. Therefore, in either case Theorem 2.6 implies $\operatorname{Tor}^R_j(k, k) = 0$ for $j \gg 0$ as desired. □

Based on Theorem 3.11 (3), we come up with the following question: if $(R, \mathfrak{m}, k)$ is a noetherian local domain of characteristic 0 and $\operatorname{Tor}^R_i(R^+, k) = 0$ for some $i \geq 1$, then is R regular?

References

[岩永佐藤] 岩永恭雄, 佐藤眞久, 環と加群のホモロジー代数的理論, 日本評論社, 2002.

[AL08] I. M. Aberbach, J. Li, *Asymptotic vanishing conditions which force regularity in local rings of prime characteristic*, Math. Res. Lett. **15**(4), 815–820.

[AJL97] L. Alonso Tarrío, A. Jeremías López, J. Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30**(1), (1997), 1–39.

[AF91] L. L. Avramov, H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71**, (1991), 129–155.

[Bha18] B. Bhatt, *On the direct summand conjecture and its derived variant*, Invent. Math. **212**(2), (2018), 297–317.

[BIM19] B. Bhatt, S. B. Iyengar, L. Ma, *Regular rings and perfect(oid) algebras*, Comm. Algebra **47**(6), (2019), 2367–2383.

[BMS1] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128**, (2018), 219–397.

[BouAC89] N. Bourbaki, *Algèbre commutative*, Chapitres 8 et 9, Springer, 2006.

[BH] W. Bruns, J. Herzog, *Cohen–Macaulay rings*, revised edition, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[CIM19] L. Christensen, S. B. Iyengar, T. Marley, *Rigidity of Ext and Tor with cooefficients in residue fields of a commutative noetherian ring*, Proc. Edinb. Math. Soc. **62**(2), (2019), 305–321.

[GM92] J. P. C. Greenless, J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149**(2), 438–453.

[Har] R. Hartshorne, *Local cohomology*, Lecture Notes in Mathematics, No. 41, Springer, Berlin-New York, 1967.

[Hei02] R. C. Heitmann, *The direct summand conjecture in dimension three*, Ann. Math. **156**(2), (2002), 695–712.

[Hei05] R. C. Heitmann, *Extended plus closure and colon-capturing*, J. Algebra **293**(2), (2005), 407–426.

[HH92] M. Hochster, C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. Math. **135**(1), (1992), 53–89.

[Kun69] E. Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. **91**, (1969), 772–784.

[Kun76] E. Kunz, *On noetherian rings of characteristic p*, Amer. J. Math. **98**, (1976), 999–1013.

[Mat2] H. Matsumura, *Commutative ring theory*. Translated from the Japanese by M. Reid. Second edition, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1989.

[Sch03] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92**(2), 161–180.

[Sch21] P. Schenzel, *About proregular sequences and an application to prisms*, Comm. Algebra **49**, (2021), 4687–4698.

[SS] P. Schenzel, A.-M. Simon, *Completion, Čech and Local Homology and Cohomology*, Springer Monographs in Mathematics, Springer, Cham, 2018.

[Shi16] K. Shimomoto, *An application of the almost purity theorem to the homological conjectures*, J. Pure Appl. Algebra **220**(2), (2016), 621–632.

[Sta] The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu, 2023.